XII. *On Symbolic Forms derived from the Conception of the Translation of a Directed Magnitude.* *By the Rev.* M. O'Brien, *M.A., Late Fellow of Caius College, Cambridge, and Professor of Natural Philosophy and Astronomy in King's College, London.*

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PART I.

GENERAL INVESTIGATION OF THE SYMBOLIC FORMS.

(1.) There can be no doubt, that time and ingenuity have been often wasted in devising systems of notation, and new methods of algebraical representation, which have never proved of any service in advancing the cause of science. It is not surprising, therefore, that symbolical innovations, if they have not the strongest and most obvious reasons to recommend them, are generally received with little favour by mathematicians. At the same time, it must not be forgotten, that the mind has wonderfully enlarged its powers of research by the symbolization of its abstract conceptions, and that the various additions which have been made, from time to time, to mathematical notation, have contributed largely to the progress of physical investigation; witness, for instance, the applications of the negative sign, indices, logarithms, coordinate equations, the differential algorithm, &c.

A new notation, or a new application of an old notation, ought, in all cases, to be called for by some want in science, that is, by the existence of some important and often occurring conception for which there is no adequate, or at least no sufficiently general mode of representation. It should be neither artificial nor complicated, but natural and simple: it should also be based on principles of established authority, and framed according to allowed precedents. And, lastly, it should be capable of something more than mere elementary applications, and be recommended by its utility in the higher and more abstruse branches of science.

With these cautions before me, and on these grounds, I venture, in the present paper, to propose a new use of an old notation, which appears to me to supply a want of considerable importance, as I hope to show by the remarkable simplifications which it introduces into many difficult investigations. There is an operation, if I may so call it, of constant occurrence in Geometry and Physics, which consists in the *translation of a directed magnitude*, that is, the parallel motion of a magnitude possessing the property of direction, such, for example, as a force, a velocity, traced line, or the like. This translation, as it may be easily shown, is always an operation of the kind called "*distributive*," and from its distributiveness all its properties in a great measure follow as necessary consequences. Now this is a fact of great importance to be borne in mind, a thing to which the truly mathematical adage "*when found make a note of*" fully applies. There is no notation actually in use for this purpose; but there are two nearly obsolete signs, which by an easy and natural generalization, and according to the most approved rules of mathematical interpretation, might be made to serve effectually as representatives of the effects produced generally by the translation of a directed magnitude. My design, in what follows, is to show this, and establish the laws according to which these signs are to be used in their enlarged signification. Afterwards I shall endeavour to justify the proposed innovation, if such it is to be considered, by showing its utility in a variety of cases.

(2.) *Instances which suggest the proposed symbolization.*—There are three elementary conceptions which have suggested to my mind the principles which it is the object of this paper to develope, and they will serve here as means of introducing the subject, and furnish the best foundation to build upon. They are the following:—1st. *The generation of surface by the parallel motion of a right line,* of which the simplest instance is a parallelogram ABCD supposed to be generated by the motion of AB, parallel to itself, along AD.—2ndly. *The effect produced on a rigid body by the translation of a force acting upon it* from one point A to another B, the direction of the force remaining unaltered; which effect, as is well known, consists in that peculiar tendency to motion that results from the action of the couple composed of the force at B, and a force equal and opposite to the original force at A.—3rdly. *The effect produced by the translation of a force resulting from the actual motion of its point of application*; which effect is now usually designated by the term *work*. If the point of application be supposed to describe the path ADB, the force all the time acting parallel to its original direction, a certain amount of work is accumulated in consequence of the translation of the force, and this is the effect I allude to.

Fig. 1.

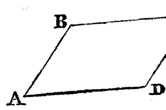


Fig. 2.

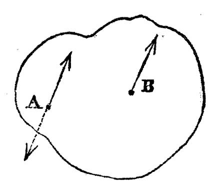
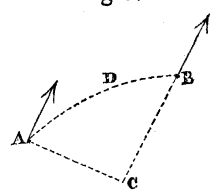


Fig 3.



Now in each of these cases the conception in the mind is that of *the effect produced by the translation of a directed magnitude*; and what is worthy of special remark is this, that in each of these cases the effect alluded to is represented by the *product of two factors, one* being *the translated magnitude,* and *the other the amount of translation* it undergoes. Thus, in the case shown by figure 1, the surface generated is denoted by the product of AB into the perpendicular or *lateral* distance between AB and CD; of which, AB is the translated magnitude, and the perpendicular the amount of translation that takes place *laterally*. In the case shown by figure 2, the effect is also represented by the product of the translated magnitude, that is, the force, into the amount of *lateral* translation. In the case shown by figure 3, if we suppose BC to be the direction of the force produced backward to meet a perpendicular AC let fall from A upon it, the effect produced, that is, the work accumulated, is represented by the product of the translated magnitude (the force, namely,) into the amount of translation which takes place, not laterally, as before, but *longitudinally*, that is, *along* the direction of the force, which longitudinal translation is manifestly CB.

(3.) From these three instances the idea naturally arises of *some necessary connection* between the translation of a directed magnitude and the product of the two factors, the magnitude translated and the amount of translation; or, to say the least, there appears to be some ground for conceiving that the product in question may be the *proper form of notation* for representing the translation. And, secondly, the necessity of *distinguishing* between *lateral* and *longitudinal* translation is clearly indicated, inasmuch as the longitudinal effect is zero in the first two cases, while the lateral effect is zero in the third case. Taking my clue from these suggestions, I shall now proceed to explain my proposed method of notation; observing, that my object is to make it as general as possible consistently with definiteness and utility, and that, for this reason, I shall employ all the generalizations of the elementary algebraical signs which are now admitted by mathematicians.

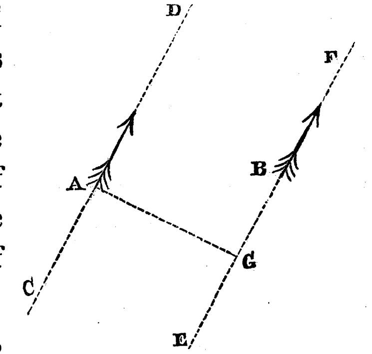
I shall also adopt, to a certain extent, the views of Symbolical Algebra taken by the late Mr. Gregory, and published by him in several papers, but especially in one read before the Royal Society of Edinburgh on the Foundations of Algebra. I may observe, however, that my proposed method of notation does not assume the correctness of these views, and might be enunciated independently of them; but they appear to my mind to form the most satisfactory theory of Symbolical Algebra.

I. Preliminary Definitions, Statements, etc.

(4.) *Directed Magnitude.*—I use this term to denote any of those magnitudes which we represent graphically by *arrows*; remarking that an arrow represents *three* things, viz. an *origin* or *point of application*, marked by its *feather-extremity*; a particular *magnitude* represented by its *length*; and a particular *direction* shown by the *barb*.

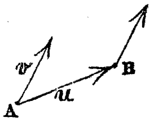
(5.) *Translation, Lateral and Longitudinal.*—"Translation" is the term employed to denote that peculiar and simplest change of position of a rigid body which consists in the parallel and equal motion of all its component particles. I shall use the same term to denote a change of position of a directed magnitude without change of direction, as is shown in fig. 4. The translation therefore of a directed magnitude consists in simple alteration of "*origin*," as from A to B in the figure.

Fig. 4.



But this alteration of origin is of a *twofold* nature, being partly *lateral* and partly *longitudinal*. If CD be the indefinite line of direction in which the arrow at A lies, and EF that in which the arrow at B lies, the translation from A to B consists of two distinct motions; namely, the shifting of the line of direction from CD to EF, and the shifting of the origin along the line of direction through a space amounting to GB. The former I shall call *lateral* and the latter *longitudinal* translation.

Fig. 5.



If v denote a directed magnitude which is supposed to be translated from A to B, and if u denote the line AB, I shall speak of the translation as *that of v along u*; a proper mode of expression, because every point of the representative arrow v undergoes a motion equivalent in magnitude and direction to u. The translation of v along u is lateral when the angle at A is 90°, and longitudinal when 0°.

(6.) *Distributiveness.*—If $f(x)$ be a function of x which possesses the property expressed by the equation

$$f(x)+f(x')=f(x+x'),$$

it is said to be a *distributive* function of x. If x be any *number* positive, negative, integral or fractional, it may be shown from this equation that

$$f(x)=\mathrm{C}x,$$

C denoting a quantity independent of x, namely the value of $f(x)$ when $x=1$. If x be not a number, but some symbol, whether of specific quantity or operation, the notation Cx has no meaning recognised in ordinary algebra. Hence, following the well-known precedent of indices*, we may generalize the meaning of Cx by assuming it to be the *symbolical form* for denoting *every function, $f(x)$, which is distributive.*

If we further suppose, that $f(x)$, and therefore C, is a distributive function of another independent variable y, we shall find that

$$\mathrm{C}=\mathrm{C}'y \quad \text{and} \therefore \mathrm{C}x=\mathrm{C}'xy.$$

Thus we may, by the same process of generalization, assume $\mathrm{C}'xy$ to be the *symbolical form* for denoting *every function* of x and y *which is distributive* with regard to both x and y. C′ here is manifestly the value of the function when $x=1$ and $y=1$. Now if we adopt C′ to be the unit of the function, as, in fact, we do in many cases of ordinary products, the symbolic form for denoting the function becomes simply xy.

(7.) This appears to me to be the simplest and best method of defining the notation xy in Symbolical Algebra; though I need not avail myself of it here as it is not necessary for my purpose. All I require is some simple notation for denoting a distributive function of two variables; for, as I hope to show, this distributiveness is a characteristic of great importance to be distinctly "noted" in the case of the translation of a directed magnitude. Now there are three different forms in which a product is written in ordinary algebra, viz. xy $x.y$ and $x\times y$: of these, the latter two are now seldom used, and there is no necessity whatever for this redundancy of

* a^x has no meaning, according to its original definition, except x be a positive integer: but we give it a meaning by defining a to be the notation for every function $f(x)$ which possesses the property $f(x)f(y)=f(x+y)$.

forms for denoting the same thing. Instead, therefore, of inventing new symbols for representing distributive functions, I shall venture to appropriate the almost obsolete forms $x.y$ and $x \times y$ to the purpose. At the same time it must be borne in mind, that, according to the views of many eminent mathematicians, the product of x and y in Symbolical Algebra may be defined to be *any* distributive function of x and y, and thus the appropriation of $x.y$ and $x \times y$ here proposed is nothing more than a legitimate application of these forms.

(8.) I shall therefore assume $x.y$ and $x \times y$ to be symbolical forms for denoting distributive functions of x and y; in other words, I shall consider $x.y$ and $x \times y$ to be completely defined by the equations

$$x.y+x'.y=(x+x').y \qquad x\times y+x'\times y=(x+x')\times y$$
$$x.y+x.y'=x.(y+y') \qquad x\times y+x\times y'=x\times(y+y'),$$

just as the symbolic form a^m is completely defined by the equation $a^m a^n = a^{m+n}$.

(9.) Whether $x.y$ and $x \times y$ are "*Commutative*" functions of x and y, i.e. whether $x.y=y.x$, and $x\times y=y\times x$, does not appear from these defining equations, and therefore it must be decided by the particular nature of the quantity or operation which each of these forms is assumed to represent.

(10.) *Signification of the sign* +. In Symbolical Algebra the sign + may be regarded as simply an abbreviation for the words "*together with*," and thus $u+v$ means simply u "together with" v, or u and v "*put together*." Now these words "*together with*" may be taken in a great variety of senses, as the following examples taken from ordinary algebra show, viz.

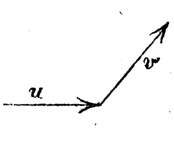
$$3+5=8, \qquad 3£+5s.=780d., \qquad 3+4\sqrt{-1}=(2+\sqrt{-1})^2,$$
$$5 \text{ miles east}+5 \text{ miles west}=0, \text{ \&c. \&c.}$$

In the first example + means a "*putting together*" by simple numerical addition; in the second, a "*putting together*" of certain pieces of gold and silver, with reference to a certain conventional value set on them; in the third, a mere symbolical "putting together;" and so on. Hence it is clear that in using the sign + as an abbreviation of the words "*together with*," the precise nature of the "*putting together*" is supposed to be understood in each case. I shall therefore define the notation, $u+v$, to mean, u *and* v *put together in a sense supposed to be understood.*

Now in some cases it is very important that the precise nature of the "*putting together*" denoted by the sign + should be clearly understood, and therefore distinctly specified. This it will be necessary for me to do here with reference to two remarkable significations which have been given to the sign +.

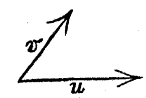
(11.) The first is that signification given to + in Symbolical Geometry. If u and v denote two lines of certain magnitudes and drawn in certain directions, then $u+v$ is assumed to denote u and v *put together* as in the figure 6; that is, the *beginning-point* or *origin* of v coinciding with the *end-point* (if I may so use the words) of u. The second signification

Fig. 6.



is that given to + in Symbolical Mechanics. If u and v denote two forces of certain magnitudes and directions, $u+v$ is assumed to denote u and v *put together* as in figure 7; that is, the *origin* of v coinciding, not with the *end-point*, as before, but with the *origin* of u. The distinction I allude to here is of considerable importance, and requires to be very closely attended to in applying lines to represent forces generally, as will appear. It might be well to distinguish these two significations of + by appropriate terms. I cannot think of any better words, for the purpose, than the two, "*successive*," and "*simultaneous*;" the *putting together* in fig. 6 is manifestly effected by tracing the two lines *in immediate succession*, while that in fig. 7 is a *simultaneous application at the same origin*. I shall therefore call the *putting together* in fig. 6 *successive addition*, and that in fig. 7 *simultaneous addition*.

Fig. 7.



(12.) *Signification of the sign* =. Like +, the sign = denotes *equivalence in a certain sense supposed to be understood*; thus in the example, $3£+5s.=780d.$, it denotes equivalence as regards the conventional value of certain coins. In Symbolical Geometry = has reference to the *change of position of the tracing-point* by which lines are supposed to be drawn. Thus if u, v, w denote three traced lines, the equation, $u+v=w$, means, that the tracing of $u+v$ is the same thing as the tracing of w, *so far as the change of position of the tracing-point is concerned.* In this sense it is clear, that w must be the third side of the triangle in fig. 8. In Symbolical Mechanics = has reference to *mechanical effect.* Thus if u, v, w be three forces, the equation, $u+v=w$, means, that *the mechanical effect of* u+v *is the same as that of* w; in other words, it means, that w is the *resultant* of u and v.

Fig. 8.

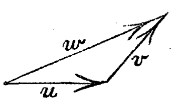
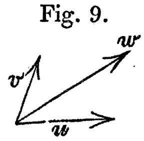


Fig. 9.



(13.) *Representation of Forces by Lines.* The suitability of lines to represent forces is obvious enough in ordinary Mechanics, where + has the signification of mere numerical addition; but when we come to Symbolical Mechanics this suitability is no longer a thing to be assumed. A little consideration will show that the question, "Can we assume lines to represent forces generally?" may be stated symbolically as follows, viz. If the lines u and v respectively represent the forces U and V, in magnitude and direction, will $u+v$ also represent U+V in magnitude and direction? if not, the graphical mode of representation becomes inadmissible symbolically. Now, it is clear, by reference to figures 8 and 9, that this question amounts to asking, whether the *Parallelogram of Forces* is true or not? for the peculiar signification of + in Symbolical Geometry makes $u+v$ denote the diagonal of the parallelogram constructed on u and v as sides; whereas the Mechanical signification of + in U+V makes it denote the resultant of U and V.

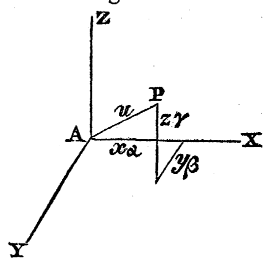
Hence it follows that the *general* representation of forces by lines *assumes* the truth of the Parallelogram of Forces as a necessary condition; and, consequently, any symbolical proof of the Parallelogram of Forces which assumes that lines may be taken generally as representatives of forces, amounts to reasoning in a circle.

My object here, however, in the remarks just made, is to point out the importance of distinctly marking the two significations of the sign $+$.

(14.) In all cases that I shall be concerned with, *successive* and *simultaneous* addition are virtually equivalent, so far as the representation of directed magnitudes by lines is concerned. As regards the *statical effect of forces*, the Parallelogram of Forces shows this: as regards the *dynamical effect*, the Second Law of Motion (I mean NEWTON's 2nd Law) does the same. As regards *velocities* and *displacements* the thing is obvious.

(15.) *Directed Units.* I shall call an arrow of a unity of length (whether it represents a traced line, a force, a velocity, or any other kind of directed magnitude) a "*directed unit.*" I shall always reserve the letters α, β, γ to denote a set of three directed units *at right angles to each other*. Hence, if AX, AY, AZ be three rectangular axes to which α, β, γ are respectively parallel; and if x, y, z denote numerically the three coordinates of any point P; $x\alpha$, $y\beta$, $z\gamma$ will be the symbols representing these three coordinates *in magnitude and direction*, inasmuch as $x\alpha$ means x directed units *put together* by *successive* or *geometrical* addition, all in the direction parallel to AX; and so also as regards $y\beta$ and $z\gamma$. Also if u be taken to denote the line AP in magnitude and direction, we have, by *successive* addition,

Fig. 10.



$$u = x\alpha + y\beta + z\gamma.$$

The point P is often called *the point* (xyz), I may therefore speak of it as *the point* (u), inasmuch as u completely defines its position.

If X, Y, Z denote *numerically* three forces parallel to AX, AY, AZ, it is clear that their complete symbolical representatives are $X\alpha$, $Y\beta$, $Z\gamma$. Also, if U denote the resultant of these three forces, we have

$$U = X\alpha + Y\beta + Z\gamma.$$

But here $+$ denotes *simultaneous* addition: we may, however, assuming the truth of the Parallelogram of Forces, regard it as the *successive* $+$, if we please.

(16.) As just observed, I shall always suppose α, β, γ to be a set of *three rectangular directed units*; I shall suppose the same also as regards α', β', γ'; α'', β'', γ'', &c., using the dashes to denote different sets of directed units; but the three in each set are always assumed to be at right angles to each other, unless the contrary be specified.

In speaking of lines as regards magnitude and direction, I shall always use the word "*direction*" as equivalent to "directed unit;" thus I shall call α the "*direction*" of the line $x\alpha$. The *complete symbol* of a line may be therefore described as *its direction multiplied by its magnitude.*

(17.) If r denote the magnitude, and α' the direction of u, $u = r\alpha'$, and therefore, putting for u its value above, we find

$$\alpha' = \frac{x}{r}\alpha + \frac{y}{r}\beta + \frac{z}{r}\gamma,$$

or

$$\alpha' = a\alpha + b\beta + c\gamma,$$

where a, b and c denote the cosines of the angles which α' makes with α, β and γ respectively.

If α' lies in the plane $(\alpha\beta)$, and if θ denote the angle which α' makes with α, this expression becomes

$$\alpha'=\alpha\cos\theta+\beta\sin\theta.$$

(18.) According to the principles of Symbolical Algebra, we have

$$\beta=-^{\frac{1}{2}}\alpha,\qquad \gamma=-^{\frac{1}{2}}\beta,\qquad \alpha=-^{\frac{1}{2}}\gamma,\qquad (-^{\frac{1}{2}}=\sqrt{-1}).$$

But it is to be remembered that the sign $-^{\frac{1}{2}}$ here does not denote the same identical operation in these three cases; nor is it necessary that it should, any more than the sign $-$. The true state of the case is this, that $(-^{\frac{1}{2}})\alpha$ is defined by the equation

$$(-^{\frac{1}{2}})(-^{\frac{1}{2}})\alpha=-\alpha;$$

and the general solution of this equation is

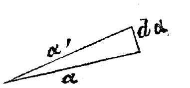
$$-^{\frac{1}{2}}\alpha=\beta\cos\theta+\gamma\sin\theta,$$

where θ is perfectly arbitrary. Consequently the extraction of the square root of $-$ gives, not simply two values, positive and negative, as in ordinary extractions of the square root, but an infinite number of values, namely the *whole circle* of directed units at right angles to α.

I shall have no occasion to make any use of the sign $-^{\frac{1}{2}}$, or any reference to the connections just given between α, β and γ, except in some future applications of my method, chiefly in Geometry. The statement just made is intended to show what α, β, γ are with reference to the square roots of $-$ (or -1), and to point out distinctly that α, β, γ are *not* supposed to be *square roots of unity*, but merely *direction-units.*

(19.) *Remarkable signification of $d\alpha$, $d\beta$, $d\gamma$.* This signification I pointed out and made use of in a paper read before the Cambridge Philosophical Society (Nov. 1846), and it may be briefly stated here for the purpose of reference in certain applications of the present method. If α and α' be two directed-units at an indefinitely small angle to each other, we have $\alpha'-\alpha=d\alpha$; but $\alpha'-\alpha$ is the line joining the extremities of α and α', and this line is at right angles to α and α' (ultimately), because α and α' are lines of equal length. Hence $d\alpha$ is the expression for an indefinitely small line *at right angles to α.*

Fig. 11.



This signification of $d\alpha$ is one of great importance in Symbolical Geometry and Mechanics: thus for example, if α, β, γ denote direction-units *fixed* in a rigid body, the angular velocities of the rigid body are represented, in magnitude and direction, by

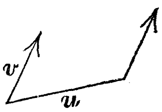
$$\frac{d\alpha}{dt},\quad \frac{d\beta}{dt},\quad \frac{d\gamma}{dt},$$

inasmuch as $d\alpha$, $d\beta$, $d\gamma$ represent, in magnitude and direction, the small angles described in the time t by the extremities of these three direction-units. Of course I mean by the word "*angle*," here, the circular arc which measures it. The importance of this signification of $d\alpha$, $d\beta$, $d\gamma$ will be manifest in many parts of what follows.

II. Symbolical Representation of the two Effects produced by the Translation of a Directed Magnitude.

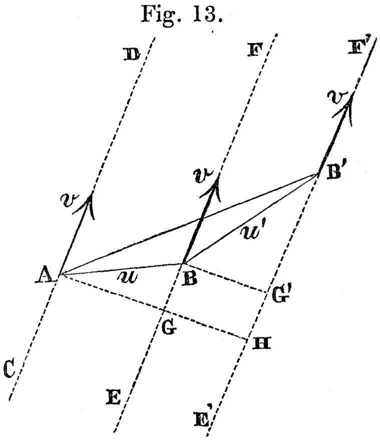
(20.) *The Effects symbolized.*—It has been shown that the translation of v *along* u is, generally, of a twofold nature, partly *lateral* and partly *longitudinal*; it is my object to express symbolically the *effects produced*, whether they are geometrical or mechanical effects, by the two kinds of translation. The effect produced by the lateral part of the translation of v along u I shall call *the lateral effect*, and that produced by the longitudinal part, *the longitudinal effect*.

Fig. 12.



(21.) *The two Effects are, each, Distributive Functions of* u *and* v. Let $f(u, v)$ denote the *lateral effect* of the translation of v along u; let u and u' represent the lines AB and BB′; produce the arrows (v) both ways indefinitely to show the *lines of direction* in which v lies in the three parallel positions at A, B, and B′; draw AGH and BG′ at right angles to these parallel lines of direction (CD, EF, E′F′). Observe, that v, u and u' are not necessarily in the same plane.

Fig. 13.



Now, as assumed, $f(u, v)$ denotes the effect produced by the shifting of the line CD to the parallel position EF; $f(u', v)$ denotes that by a farther shifting, namely from EF to the parallel position E′F′: which two shiftings "*put together*" come to the same thing as one shifting from CD to E′F′. Now since AB′ is represented by $u+u'$, the effect of this last-mentioned shifting is denoted by $f(u+u', v)$: we have therefore

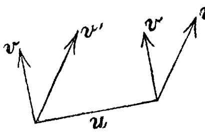
$$f(u, v)+f(u', v)=f(u+u', v),$$

that is, the *lateral effect* of the translation of v along u is a *distributive function as regards* u.

In precisely the same way it may be shown, that the *longitudinal effect* is also a distributive function as regards u. For it is manifest that the three translations, viz. that along u, that along u', and that along $u+u'$, amount respectively to GB, G′B′, and HB′, as regards longitudinal effect, and we have HB′=GB+G′B′. Whence the conclusion is evident.

It remains to show that both effects are distributive functions *as regards* v *also*; and this is immediately obvious: for the translation of v along u "*together with*" that of v' along u, is the same thing as the translation of $v+v'$ along u; and thus, whether f denote the lateral or longitudinal effect, we have

Fig. 14.



$$f(u, v)+f(u, v')=f(u, v+v').$$

Both effects therefore are distributive functions with respect to v as well as u.

(22.) It is important to observe, here, that the $+$ in $u+u'$ denotes *successive*, while that in $v+v'$ denotes *simultaneous addition.*

(23.) *Notation adopted to represent the two effects.* I have stated above the reasons why $u.v$ and $u\times v$ may be appropriated to denote, simply, *any* distributive functions of u and v. I have here shown the existence of two such functions, very important to be "noted" symbolically, and to be distinguished from each other. I shall therefore venture farther to employ the notation $u.v$ *exclusively* for the purpose of representing the *lateral effect* of the translation of v along u, and the notation $u\times v$ *exclusively* to represent the *longitudinal effect.*

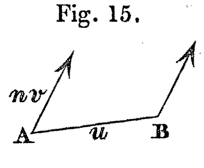
(24.) As regards the *order of the factors*, I shall always suppose that the *second factor* is the translated magnitude, and the *first factor* the line along which it is translated.

(25.) In using these notations I am not warranted to attribute to them, without proof, any property of an ordinary product, except its distributiveness: for example, I must not put $u.v=v.u$, without investigating whether this equation holds as regards the effects represented by $u.v$ and $v.u$. Nor again, if m and n be any numbers, can I, without proof, put $(mu).(nv)=mn(u.v)$. These points I shall now consider.

(26.) *May Numerical Coefficients, occurring in* u.v *or* u$\times$v, *be brought out and incorporated by actual multiplication?*—Supposing m and n to denote pure numbers, may we put $(mu).(nv)=mn(u.v)$, and $(mu)\times(nv)=mn(u\times v)$? Or, to express the question in words, is the effect produced by the translation of nv along mu equivalent to mn times the effect of the translation of v along u? It is very important to bear in mind, as regards this question, that nv means $v+v+v+$ &c. "*put together*" by *simultaneous* addition; while mu means $u+u+u+$ &c. "*put together*" by *successive* addition (see art. 22). Hence it will not be difficult to show that $u.(nv)=n(u.v)$ in virtue of the distributive property; but, that some additional consideration is requisite to determine whether $(mu).v=m(u.v)$.

(27.) *First,* as regards $u.(nv)$. The $n\,v^s$ here have the same origin A, and they are translated simultaneously from A to B. This translation is manifestly the same as if each v were translated separately from A to B: and thus it follows that the translation of nv along u is the same thing as n translations of v along u; or, in symbols,

Fig. 15.

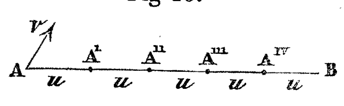


$$u.(nv)=n(u.v), \text{ and } u\times(nv)=n(u\times v).$$

Indeed this is nothing more than a re-assertion of the distributive nature of the translation of v along u, as regards v.

(28.) *Secondly.* In the expression $(mu).v$ the u^s have *not* the same origin, but are "*put together*" *successively,* as is represented in fig. 16, making up the line AB (supposing, for a moment, that $m=5$). It is clear, then, that $(5u).v$ means a translation of v from A to B, while 5 $(u.v)$ means five translations of v from A to A$'$. Hence, before we can decide whether $(mu).v=m(u.v)$, we

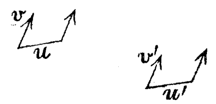
Fig 16.



must determine whether the translations of v from A to A$'$, from A$'$ to A$''$, from A$''$ to A$'''$, &c. are equivalent to each other *in effect*; for, if they are, it makes no difference *in effect* whether we repeat the translation of v from A to A$'$ five times, or simply translate v from A to A$'$, from A$'$ to A$''$, from A$''$ to A$'''$, and so on to B. The question then comes to this—Are we to regard translations as equivalent to each other, in effect, when the magnitudes translated and the lines along which they are translated are respectively equivalent to each other, whether the translations take place *in the same part of space or not*?

(29.) *Fundamental Assumption.*—I am thus led to make the following *Assumption* the basis of my proposed method; viz.—*That parallel and equal translations of parallel and equal magnitudes are equivalent to each other both as regards the lateral and longitudinal effects*; or, symbolically, if u' and v' be respectively parallel and equal to u and v, then

Fig. 17.



$$u'.v'=u.v, \text{ and } u'\times v'=u\times v.$$

This assumption holds true, manifestly, in each of the three *suggesting cases* from which I have taken my start (see art. 2), and I am therefore justified in adopting it, *with the understanding*, of course, that it be shown to hold true, or tacitly admitted, in all cases to which the notation may be applied; or else, should the occasion require it, be abandoned, and, with it, the property expressed by the equation $(mu).v=m(u.v)$.

(30.) Returning to fig. 16, we have, by the Assumption just made,

$$\text{AA}'.v=\text{A}'\text{A}''.v=\text{A}''\text{A}'''.v=\text{\&c. \&c.};$$
$$\therefore m(u.v)=m(\text{AA}'.v)=\text{AA}'.v+\text{A}'\text{A}''.v+\text{A}''\text{A}'''.v+\text{\&c.}$$
$$=(\text{AA}'+\text{A}'\text{A}''+\text{A}''\text{A}'''+\text{\&c.}).v$$
$$=(mu).v.$$

And generally, by what has been proved, we have

$$(mu).(nv)=n\{(mu).v\}=mn(u.v);$$

and, similarly, $$(mu)\times(nv)=mn(u\times v).$$

It appears thus that *numerical coefficients*, occurring in the symbolic forms $u.v$ and $u\times v$, may always be *brought out* and incorporated by actual multiplication*.

(31.) *May the order of the factors* u *and* v *in the symbolic forms* u.v *and* u$\times$v *be changed, or not?*—*First*, as regards $u.v$, may we put $u.v=v.u$? Here I may repeat that the second factor always denotes the translated magnitude, or rather, the representative arrow. Thus the question is—as regards lateral effect, is the translation of the magnitude represented by the *arrow* v along the *line* u equivalent to the translation of that represented by the *arrow* u along the *line* v. This is easily decided as follows.

(32.) It is clear that the *lateral effect* of the translation of a magnitude *in its own*

* It may be shown, in the usual way, that this is true also when m and n are *fractional* or *negative* numbers.

direction is zero: it follows therefore that $u.u=0$, $v.v=0$; also that

$$(u+v).(u+v)=0,$$

$$\therefore u.u+u.v+v.u+v.v=0,$$

$$\therefore u.v+v.u=0,$$

$$\text{or } v.u=-u.v.$$

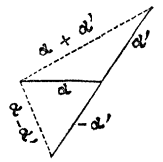
It appears then that $u.v$ and $v.u$ are *equivalent as regards magnitude* but *opposite in sign.*

(33.) *Secondly,* as regards $u\times v$, may we put $v\times u=u\times v$? This question is determined by observing that the *longitudinal effect* of the translation of a magnitude *at right angles to its direction is zero,* as follows.

Let $u=m\alpha$, $v=m'\alpha'$, α and α' being the *directions* (directed units), and m and m' the magnitudes of u and v. Then, by article 30,

$$u\times v=mn(\alpha\times\alpha') \text{ and } v\times u=mn\ (\alpha'.\alpha).$$

Fig. 18.



Now it is clear, from figure 18, that $\alpha+\alpha'$ and $\alpha-\alpha'$ are lines at right angles to each other; therefore

$$(\alpha+\alpha')\times(\alpha-\alpha')=0=(\alpha-\alpha')\times(\alpha+\alpha');$$

therefore, omitting common terms, we find

$$-\alpha\times\alpha'+\alpha'\times\alpha=\alpha\times\alpha'-\alpha'\times\alpha,$$

$$\therefore \alpha\times\alpha'=\alpha'\times\alpha.$$

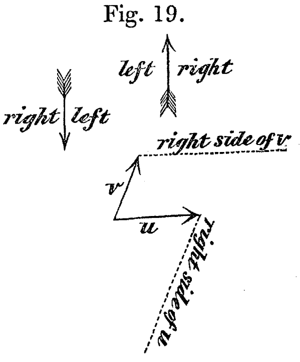
And thus it follows that

$$u\times v=v\times u.$$

It appears then that $u\times v$ and $v\times u$ are *equivalent as regards both magnitude and sign.*

(34.) Thus $u.v$ is *commutative with change of sign,* while $u\times v$ is *simply commutative.* The reason of the change of sign in the former may be easily interpreted as follows. An arrow has two distinct *sides,* which, for the sake of fixing ideas, I may call *right* and *left,* and which may be defined by supposing that I stand *on* the plane of the paper *looking in the direction of the arrow.* Now, referring to fig. 19, it is clear that the translation of v along u is *laterally* a motion *to the right side,* while that of u along v is *to the left,* the translated magnitude in both cases being that with reference to which I speak of right and left. Thus the meaning of the equation $v.u=-u.v$ is obvious.

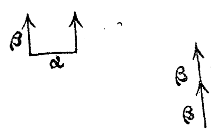
Fig. 19.



III. Measurement and Summation of Translations.

(35.) *Units of Translation.*—I shall take the translation of a unit *along a perpendicular unit* to be a *unit of lateral translation*; and the translation of a unit *along itself* to be a *unit of longitudinal translation.* Thus (see art. 16) $\alpha.\beta$, $\beta.\gamma$, $\alpha'.\beta'$, &c. are units of lateral translation; and $\alpha\times\alpha$, $\beta\times\beta$, $\alpha'\times\alpha'$, &c. are units of longitudinal translation.

Fig. 20.



(36.) *All Units of Longitudinal Translation are equivalent to each other.* For let α and α' denote any two directed units whatever; then, as above,

$$(\alpha+\alpha')\times(\alpha-\alpha')=0;$$

wherefore, since $\alpha\times\alpha'=\alpha'\times\alpha$, we have

$$\alpha\times\alpha=\alpha'\times\alpha'.$$

In illustration of this result the third *suggesting instance*, that of *Mechanical Work*, may be quoted; inasmuch as work is the effect of longitudinal translation, and all units of work are equivalent to each other, no matter in what directions the working forces act.

(37.) *All Units of Lateral Translation, in the same or in parallel planes, are equivalent to each other.*—Let α, β, α', β' lie in the same plane, and let θ denote the angle which α' makes with α, and therefore that also which β' makes with β (art. 16): then (art. 17)

$$\alpha'=\alpha\cos\theta+\beta\sin\theta$$

$$\beta'=\alpha\cos\left(\theta+\frac{\pi}{2}\right)+\beta\sin\left(\theta+\frac{\pi}{2}\right);$$

$\therefore$, observing that $\alpha.\alpha=\beta.\beta=0$, and $\alpha.\beta=-\beta.\alpha$,

we have

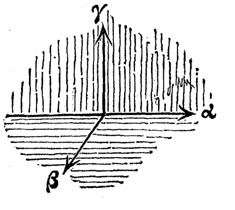
$$\alpha'.\beta'=\alpha.\alpha(\cos^2\theta+\sin^2\theta)=\alpha.\alpha.$$

Hence all units of lateral translation in the same, or in parallel planes, are equivalent to each other.

In illustration of this result, the second *suggesting instance*, that of a *Couple*, may be quoted; for all unit-couples in the same or in parallel planes are equivalent to each other.

(38.) *Directrix.*—Hence, in expressing a unit of lateral translation, it is only necessary to specify a plane parallel to that in which the translation takes place; or, what is better and immediately suggested by the theory of couples, it is only necessary to specify *a line at right angles to the plane of translation.* Such a line I shall, however, designate by the word "*directrix*," not *axis*; because there is no idea of rotation involved in the present theory, translation being a kind of motion essentially different from rotation. I shall assume γ to be the directrix of $\alpha.\beta$ and of all units of lateral translation in planes at right angles to γ; and, generally, I shall define the directrix of any unit of lateral translation to be a directed unit at right angles to the plane of that translation.

Fig. 21.



(39.) But, since $\alpha.\beta=-\beta.\alpha$, it is necessary to distinguish positive from negative translations; and this may be done by giving an appropriate sign to the directrix. I shall therefore assume generally, that $m\gamma$ is the directrix of $m\,\alpha.\beta$, m being any number positive or negative. Thus $-\gamma$ will be the directrix of $-\alpha.\beta$, that is, of $\beta.\alpha$. And, hence, I may adopt the following criterion of sign. I shall suppose myself standing at right angles to the plane of translation in such a position that the translated magnitude points to the right, while I face the direction in which the translation takes place; and then I shall take an arrow pointing *from foot to head* as the directrix. According to this criterion the letters written underneath the following translations are their respective directrices; viz.

$$\begin{array}{cccccc} \alpha.\beta & \beta.\gamma & \gamma.\alpha & \beta.\alpha & \gamma.\beta & \alpha.\gamma \\ \gamma & \alpha & \beta & -\gamma & -\alpha & -\beta \end{array}$$

(40.) *Measurement of a Translation.*—Let $u.v$, or $u\times v$, be the translation, α the direction of u, $(\alpha\beta)$ the plane of $u.v$, θ the angle which v makes with u, m and n the magnitudes of u and v; then

$$u=m\alpha,\quad v=n(\alpha\cos\theta+\beta\sin\theta);$$
$$\therefore\ u.v=mn\sin\theta(\alpha.\beta),\quad (\text{since } \alpha.\alpha=0).$$

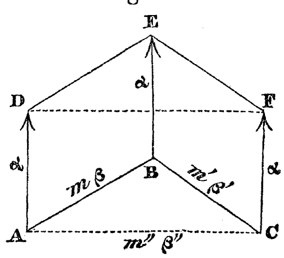
Hence there are $mn\sin\theta$ units in the translation $u.v$: $mn\sin\theta$, therefore, is the numerical magnitude of $u.v$, and its directrix is $(mn\sin\theta)\gamma$.

Again, $u\times v=mn\cos\theta(\alpha\times\alpha)$, (since $\alpha\times\beta=0$). Hence the numerical magnitude of $u\times v$ is $mn\cos\theta$.

(41.) It is worth remarking that $mn\sin\theta$, the numerical magnitude of $u.v$, is the area of the parallelogram completed on u and v as sides.

(42.) *The Directrix of the Sum of two translations is the Sum of their Directrices.*—Let the two translations be $m\alpha.\beta$ and $m'\alpha'.\beta'$, and their directrices, of course, $m\gamma$ and $m'\gamma'$; let EB be the intersection of the two planes of these translations, and take AB$=m$ and BC$=m'$, AB and BC being drawn at right angles to BE, AB in the plane of $\alpha.\beta$, and BC in the plane of $\alpha'.\beta'$.

Fig. 22.



Now, since all units of lateral translation in the same plane are equivalent to each other, I may turn α and β, α' and β' about in their respective planes, until both α and α' coincide with BE; in which case β and β' will coincide with AB and BC respectively; then AB will become $m\beta$, and BC $m'\beta'$. Let the third side of the triangle ABC be $m''\beta''$ as shown in the figure. Then the sum of the two translations is

$$m\alpha.\beta+m'\alpha'.\beta',$$

which, since $\alpha'=\alpha$, becomes

$$\alpha.(m\beta+m'\beta');$$

and this, since $m\beta+m'\beta'=m''\beta''$, becomes

$$m''\alpha.\beta''.$$

Let $m''\gamma''$ be the directrix of this: then it is clear that γ, γ' and γ'', being each at right angles to α, lie in the plane of the triangle ABC, and consequently $m\gamma$, $m'\gamma'$ and $m''\gamma''$ are the three sides of the triangle ABC, supposing it to be turned round in its plane through 90°. It follows therefore that

$$m''\gamma''=m\gamma+m'\gamma',$$

that is, the directrix of the sum of the two translations is the sum of their directrices.

(43.) Generally, it is manifest from this result, that the *Directrix of the Sum* of any number of Translations is the *Sum of their Directrices.*

It is not necessary to point out the importance of this rule as regards the *summation* of translations, nor its identity with the well-known rule in the *Theory of Couples.*

(44.) For the sake of convenience it will be worth while to employ some abbreviated mode of specifying the directrix of any translation; for this purpose I shall adopt the following notation, which, it will be found, will answer all purposes, and at the same time very distinctly mark that property in which the peculiar relation between a translation and its directrix consists. The property I allude to is the theorem just proved in article 42.

I shall employ the letter D to stand as an abbreviation for the words "*directrix of,*" and thus $D(u.v)$ will mean *the directrix of the translation* u.v. It will be borne in mind, then, that $D(u.v)$ denotes a line at right angles to the plane of $u.v$, and containing as many units of length as there are units of area in the parallelogram constructed on u and v.

(45.) *Distributiveness of the Operation thus represented.*—It is most important to notice the *distributive* nature of the symbol D. By art. 42, we have immediately

$$D(u.v)+D(u'.v')=D(u.v+u'.v');$$

whence, D denotes a distributive function.

(46.) *Consequences hence resulting.*—From the equation in art. 45 it follows, that, if m denote any numerical coefficient, positive or negative,

$$D(mu.v)=mD(u.v).$$

Again, since

$$D(u'.v')-D(u.v)=D(u'.v'-u.v),$$

we have, passing to limits,

$$d\big(D(u.v)\big)=D\big(d(u.v)\big);$$

whence also,

$$\int\big(D(u.v)\big)=D\int(u.v).$$

In short, in all operations in which differentiation and integration are concerned, D is to be regarded as if it were an ordinary constant coefficient.

Again, if we have an equation of the form

$$u.v+u'.v'+u''.v''+\&c.=0, \quad \text{(1.)}$$

there result from it

$$D(u.v)+D(u'.v')+D(u''.v'')+\&c.=0. \quad \text{(2.)}$$

And, conversely, (2.) gives (1.).

(47.) *Inverse of* D.—If w be the directrix of $u.v$, and therefore $w=D(u.v)$; I may of course, according to the true force of the index (-1), assert, that $u.v=D^{-1}w$. Thus $D^{-1}w$ comes to be an abbreviation for the words—"*the translation whose directrix is w.*"

(48.) It may be well to observe that the following relations result from what has been said, viz.

$$D(\alpha.\beta)=\gamma \qquad D(\beta.\gamma)=\alpha \qquad D(\gamma.\alpha)=\beta$$
$$D(\beta.\alpha)=-\gamma \qquad D(\gamma.\beta)=-\alpha \qquad D(\alpha.\gamma)=-\beta$$

whence

$$\alpha.\beta=D^{-1}\gamma \qquad \beta.\gamma=D^{-1}\alpha \qquad \gamma.\alpha=D^{-1}\beta$$

&c. &c. &c.

(49.) *Symbol for Units of Longitudinal Translation.*—It has been shown that these units are all equal to each other; a *unit of longitudinal translation* is therefore an *absolute constant.* No distinctive symbol is necessary, therefore, to represent these units, and it will be allowable to employ the common unit (1) for the purpose; just in the same way that we denote all units, whether they be linear, superficial, cubical, mechanical, by this common symbol. I shall therefore always represent a unit of longitudinal translation by 1; and thus put

$$\alpha\times\alpha=1, \qquad \beta\times\beta=1, \qquad \gamma\times\gamma=1\,;$$

and generally, if m and n denote the magnitudes of u and v, we have (art. 41)

$$u\times v=mn\cos\theta,$$

θ being the angle made by u and v.

(50.) Hence if α and α' be any two "*directions,*" we have

$$\alpha\times\alpha'=\text{cosine of angle made by } \alpha \text{ and } \alpha'.$$

To this may be added

$$\text{magnitude of } \alpha.\alpha'=\text{sine of same angle.}$$

(51.) *Projections represented by the lateral and longitudinal translation-products.*—It is clear from the principles just established, that, if α, β, γ denote the directions of three coordinate axes, and v any line, the *projections* of v on the three axes are, numerically,

$$\alpha\times v, \qquad \beta\times v, \qquad \gamma\times v.$$

Again, since $u\times u$ is the square of the magnitude of the line u, the projection of v on u is, numerically,

$$\frac{u\times v}{\sqrt{u\times u}},$$

which, putting for u and v the values $x\alpha+y\beta+z\gamma$, $x'\alpha+y'\beta+z'\gamma$, becomes by *longitudinal multiplication,*

$$\frac{xx'+yy'+zz'}{\sqrt{x^2+y^2+z^2}}.$$

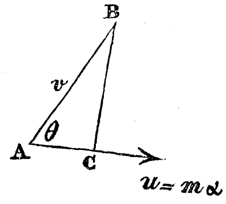
I may use the terms "*lateral* and *longitudinal multiplication*" to designate the operations denoted by $u.v$ and $u\times v$; for the word "multiplication" has quite lost its original and proper signification even in ordinary algebra.

(52.) If AB represent v, AC the projection of v on u, the magnitude and direction of u being m and α; then

$$CB=AB-AC=v-(\alpha\times v)\alpha.$$

I think CB might be advantageously called the *Complement of the Projection* of v on u, for CB added to AC makes up or *completes* v; and thus, employing the usual abbreviation, we may call CB the *coprojection* of v on u.

Fig. 23.



Hence the operation $(\alpha(\alpha\times)$ performed on v gives the *projection* of v on u, and the operation $(1-(\alpha\times\alpha)$ the *coprojection*; where α denotes the "*direction*" of u, *i. e.*

$$\frac{u}{m} \text{ or } \frac{u}{\sqrt{u\times u}}.$$

(53.) Since $\alpha.v=\alpha.(\mathrm{AC}+\mathrm{CB})=\alpha.\mathrm{CB}$, it is clear that the symbol $\alpha.v$ denotes, in magnitude, the coprojection of v on α; and it also represents the plane of projection. But $\mathrm{D}\alpha.v$ is a better symbol to use; for its magnitude is the same as that of $\alpha.v$, and, in direction, it denotes a line at right angles to the plane of projection.

(54.) *Repetition of the Operation* (Dα.). This operation is often to be performed twice in investigations, and, on this account, the following relation is important, viz.

$$(\mathrm{D}\alpha.)^2\alpha'=(\alpha\times\alpha')\alpha-\alpha'. \qquad (1.)$$

Of course $(\mathrm{D}\alpha.)^2\alpha'$ means $\mathrm{D}\alpha.(\mathrm{D}\alpha.\alpha')$.

To prove this, let β be chosen so as to lie in the plane $(\alpha\alpha')$, and let θ denote the angle which α and α' make with each other; then (by art. 17)

$$\alpha'=\alpha\cos\theta+\beta\sin\theta$$
$$\mathrm{D}\alpha.\alpha'=\gamma\sin\theta \text{ (art. 48)}$$
$$\mathrm{D}\alpha.(\mathrm{D}\alpha.\alpha')=-\beta\cos\theta$$
$$=\alpha\cos\theta-\alpha'$$

or $$(\mathrm{D}\alpha.)^2\alpha=(\alpha\times\alpha')\alpha-\alpha' \text{ (art. 50).}$$

If $u=m\alpha$ $v=n\alpha'$, we have

$$(\mathrm{D}u.)^2v=m^2n(\mathrm{D}\alpha.)^2\alpha'$$
$$=(m\alpha\times n\alpha')m\alpha-(m\alpha\times m\alpha)n\alpha',$$

or $$(\mathrm{D}u.)^2v=(u\times v)u-(u\times u)v; \qquad (2.)$$

or, if $m=1$, $$(\mathrm{D}\alpha.)^2v=(\alpha\times v)\alpha-v. \qquad (3.)$$

(55.) *Relation of the Operation* (Dα.) *to the operation* $\sqrt{-1}$ *or* $(-)^{\frac{1}{2}}$. The definition of the index $\frac{1}{2}$ in relation to operations is this. If Ω and $\Omega_{,}$ be symbols of operation, such that Ω performed twice on a quantity gives the same result as $\Omega_{,}$ once performed, then Ω is denoted by $\Omega_{,}^{\frac{1}{2}}$. Now, if we suppose that v is at right angles to α, and therefore $\alpha\times v=0$, the equation (3.) gives (what indeed is otherwise more easily shown from art. 48)

$$(\mathrm{D}\alpha.)^2v=-v,$$

wherefore

$$\mathrm{D}\alpha.=(-)^{\frac{1}{2}}.$$

In this case α is any unit line whatever at right angles to v, and therefore, in Solid Geometry, $(-)^{\frac{1}{2}}$ or $\sqrt{-1}$ has not two (as in Plane Geometry) but an infinite number of values.

(56). It is clear from the relation

$$(\mathrm{D}\alpha.)^2v=-v,$$

that (Dα.) has all the properties of the sign $\sqrt{-1}$, *provided it be performed on lines at right angles to* α. But (Dα.) is a far better sign for actual use in Solid Geometry than

$\sqrt{-1}$, because the latter is indefinite, not distinguishing what particular root of $-$ is meant; but the former is perfectly definite, inasmuch as it indicates one particular root of $-$, namely, that root which denotes rotation through $90°$ about α as axis*.

* Sir W. HAMILTON, in his System of Symbolic Geometry and Quaternions, which may be truly described as one of the most profound and beautiful theories in the whole range of abstract science, assumes the letters i, j, k to denote particular values of $\sqrt{-1}$. In a certain limited sense, the symbols $(D\alpha.)$, $(D\beta.)$, $(D\gamma.)$ are equivalent to these; for i, j, k denote rotation through $90°$ about the axes α, β, γ; and thus we have

$$\gamma=i\beta, \quad \alpha=j\gamma, \quad \beta=k\alpha;$$

and, therefore, since

$$\gamma=D\alpha.\beta, \quad \alpha=D\beta.\gamma, \quad \beta=D\gamma.\alpha,$$

it is clear that i, j, k, and $(D\alpha.)$, $(D\beta.)$, $(D\gamma.)$, so far, denote the same operations.

But the operations are different *in general*; for $i^2=-1$ *always*, but $(D\alpha.)^2$, as has been shown above, is *not* equivalent to -1, *except* when performed on lines at right angles to α.

There is one difficulty, I confess, I cannot get over in Sir W. HAMILTON'S Theory, no doubt from some misconception on my part, or from taking too narrow a view of the meaning of the sign $\sqrt{-1}$. The difficulty I allude to consists in this. Sir W. HAMILTON assumes i, j, k not only to be particular values of $\sqrt{-1}$, but also *absolute directions* (i. e. *units of direction*): in short he uses i, j, k in the same sense as α, β, γ above, and in the same sense *also* as $(D\alpha.)$, $(D\beta.)$, $(D\gamma.)$. Now my difficulty arises from my not being able to see how particular values of $\sqrt{-1}$ can denote anything (geometrically) but *change* of direction, or to perceive, that they can be used with propriety as symbols of those rectangular units of direction.

However this may be, it is important to explain the fact, that, in the results to which I have been led by the conception of translation, there are no general relations corresponding to

$$i^2=-1, \quad j^2=-1, \quad k^2=-1.$$

In my method, α, β, γ are *simple units of direction* and *nothing more*; and instead of the relations just put down, I have been led, by the conception of translation, to the following, viz.—

$$\alpha.\alpha=0, \quad \beta.\beta=0, \quad \gamma.\gamma=0$$

$$\alpha\times\alpha=1, \quad \beta\times\beta=1, \quad \gamma\times\gamma=1;$$

though, as regards the last three relations, all that I have a right to assert as a matter of necessity is, that

$$\alpha\times\alpha=\beta\times\beta=\gamma\times\gamma.$$

Also, I find

$$(D\alpha.)^2=-1, \quad (D\beta.)^2=-1, \quad (D\gamma.)^2=-1,$$

but only when performed on lines at right angles to α, β, γ respectively.

I may observe that if uv denote the product of u and v according to Sir W. HAMILTON'S Theory, it may be thus expressed in terms of my *translation-products*; viz.—

$$uv=-u\times v+Du.v.$$

Hence, since

$$Du.v=-Dv.u, \text{ and } u\times v=v\times u,$$

$$vu=-u\times v-Du.v.$$

Wherefore

$$u\times v=-\frac{1}{2}(uv+vu)$$

$$Du.v=\frac{1}{2}(uv-vu).$$

I may also observe, that, according to my method, I might put

$$uv=u\times v+u.v,$$

supposing that uv denotes the complete product of the translation of u along v, *including both* the lateral and longitudinal effects. But I cannot make any nearer approximation to the equation

$$uv=-u\times v+Du.v;$$

nor can I see that the conception of translation furnishes any interpretation of the $-$ before $u\times v$.

If, in any way, I could show that -1 was the proper value for a unit of longitudinal translation, I should have

$$\alpha\alpha=\alpha\times\alpha+\alpha.\alpha=-1.$$

(57.) The lateral translation-product is therefore well adapted to denote *rotation*. Thus (always supposing that v is at right angles to α), the symbol of a line *making an angle* θ *with* v, *and in a plane at right angles to* α, is

$$\varepsilon^{\theta D\alpha.} v,$$

which in fact is the same thing as

$$\{\cos\theta + (D\alpha.)\sin\theta\}v,$$

for cos θ and sin θ represent

$$1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4},$$

and

$$\frac{\theta}{1} - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2.3.4.5},$$

and $\varepsilon^{\theta D\alpha.}$ is the symbolic form for expressing

$$1 + \frac{(\theta D\alpha.)}{1} + \frac{(\theta D\alpha.)^2}{1.2} + ;$$

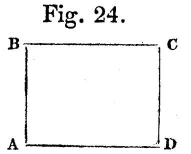
whence, since $(D\alpha.)^2 = -$ $(D\alpha.)^4 = +$ &c., the symbolic equivalence of the forms

$$\varepsilon^{\theta D\alpha.} \text{ and } \cos\theta + (D\alpha.)\sin\theta$$

is manifest. But the quantity operated upon must necessarily be at right angles to α; for, otherwise, $(D\alpha.)^2$ is not equivalent to $-$, as appears from art. 54, equation (3.).

(58.) *Concluding Remarks.*—I have now said enough, I think, to explain the nature of the proposed symbolization, and the general rules which regulate the application of the two *translation-products*. I have based the whole theory *simply and exclusively* on the conception of translation, taking my clue from the three suggesting instances, the *parallelogram*, the *couple*, and *work*. As regards the lateral effect of translation, the theory is nothing but a general development of our *geometrical* notion of multiplication; for what is a rectangle, ABCD, *considered apart from arithmetical measure*, but the effect or *product* of the translation of AB along AD? and this we represent by writing AD before or after AB. But this method of representation is clearly incomplete when we put for AB and AD their numerical representations; and why? because the special superficial unit then is omitted. For, suppose AB=3, AD=4; then, if we say that the rectangle ABCD is the product of 3 and 4, or 12, we mean, 12 *superficial units*. Now, by omitting the superficial unit in our representation, we leave out all conception of *the plane* in which the rectangle lies. All that I have done above is *to restore the superficial unit*, and determine its proper representative symbol. As regards the *longitudinal-effect* (suggested by the conception of work), it appears that all units are absolutely equivalent, and therefore may be all confounded in the common symbol of unity.

Fig. 24.



I now proceed to give Applications of the Symbolic forms* $u.v$ and $u \times v$, &c.

* I may refer here to an imperfect attempt I made in a paper read before the Cambridge Philosophical Society (Nov. 1846) to base the symbolic form $Du.v$ on the conception of perpendicularity in art. 19 above.

PART II.

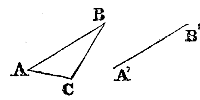
APPLICATIONS OF THE SYMBOLIC FORMS.

I. Geometrical Applications of the Symbolic Forms.

The geometrical applications which may be made of the principles and notation just explained are of great variety and importance; but, as I am anxious to dwell chiefly on physical applications, I shall only touch on this part of the subject.

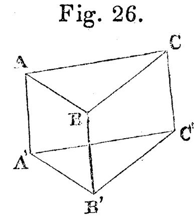
(59.) *Surface symbolically considered.*—In Symbolic Geometry the symbols of lines represent, not merely length but also direction. A line is supposed to be the geometrical effect of the motion of a tracing-point, and the equivalence of lines is considered altogether and exclusively with reference to the change which they represent in the position of the tracing-point. We consider that AB is equivalent to A′B′, when the two lines are *of equal length, parallel, and traced the same way*; for then they represent equivalent changes of position of their respective tracing-points. It is usual to employ the notation AB to denote the line AB *traced from* A *to* B, and BA to denote the same line traced from B to A; and thus AB=A′B′, but BA=−A′B′. It is clear then that equivalent lines must be, not only of equal length and parallel, but also must be traced *the same way*. Again, the equivalence being considered only with reference to the tracing-point's change of position, AC+CB is equivalent to A′B′.

Fig. 25.



Now, following this analogy, and regarding surface as the effect of the motion of a *tracing-line*, just as a line is the effect of the motion of a *tracing-point*, we may employ symbols to denote surfaces, not merely as regards their numerical area, but also with reference to the manner in which they are generated by their tracing-lines; and we may also define the equivalence of surfaces altogether and exclusively with reference to the change which they represent in the position of the tracing-line. Thus, if ABA′B′, BCB′C′, ACA′C′ be three parallelograms, and if we conceive them to be generated, respectively, by the *translations* of A′A along A′B′, B′B along B′C′, and A′A along A′C′; it is clear that

Fig. 26.



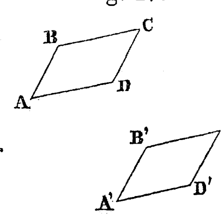
$$(ABA'B')+(BCB'C') \text{ is equivalent to } (ACA'C'),$$

just in the same sense that

$$(AB+BC) \text{ is equivalent to } (AC).$$

Again, if ABCD and A′B′C′D′ be two parallelograms, AB and AD being respectively parallel and equal to A′B′ and A′D′, we may consider these parallelograms as equivalent to each other, in the same sense exactly that the lines AB and A′B′ are regarded as equivalent. Only, just as it is necessary for the equivalence of AB and A′B′, that they should be traced *the same way* by their respective *tracing-points*, so it is necessary to the equivalence of ABCD and A′B′C′D′, that they should be traced *the same way* by their respective *tracing-lines*. Now here it is to be observed that a tracing-point is devoid of two important properties which

Fig. 27.



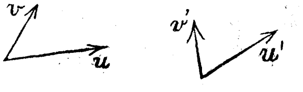
a tracing-line possesses; I mean, *direction* and *two-sidedness* (if I may so speak). An arrow lying on* a plane, not only points in a particular direction, but has *two distinct sides*, right and left (see art. 34). Hence when we speak of lines described *the same way* by their tracing-points, and parallelograms described *the same way* by their tracing-lines, the expression "*same way*" includes much more in the latter than in the former case. For example, the parallelograms generated by the translation of AB along AD, and of DC along DA, though coincident, are described *opposite ways*. Again, the parallelograms generated by the translation AB along AD, and of AD along AB, though coincident, are described opposite ways; for the former is described by a *right-side* motion, and the latter by a *left-side* (see art. 34).

(60.) *Definition of Surface by reference to Lateral Translation.*—It appears to me that the simplest way of including the considerations just alluded to in the general conception of surface requisite in Symbolical Geometry, is to define Surface by reference to the Lateral Effect of the Translation of one line along another. The *Fundamental Assumption* (article 29) is justified in this case, as appears from the remarks just made, and the peculiar relation $u.v=-v.u$ is naturally interpreted (see art. 34). I shall therefore define *Surface* in *Symbolic Geometry* to be *the Lateral Effect of the Translation of one line along another.* By "*Effect*" here I mean simple *Geometrical* effect, *i. e. change of position in space.* Also I only speak of *lateral effect*; because all notion of *longitudinal* effect is excluded by our ordinary conception of surface, and we may assume that no shifting which a line undergoes *in its own direction* can generate *surface.*

(61.) *Longitudinal Effect considered geometrically.*—But though the shifting of a line in its own direction generates no surface, it produces alteration of position; and hence it constitutes an important conception. The only difficulty, in considering the longitudinal effect *geometrically*, consists in this—How is it that all units of longitudinal translation are equivalent (see arts. 36, 37), while those of lateral translation are not, and on what principle can this difference be interpreted? The answer appears to be this: that $\alpha.\beta$ denotes the effect of translating the unit β along the perpendicular unit α; that this operation *conveys the conception of a particular plane*, and we must think of $\alpha.\beta$, $\beta.\gamma$ as different operations because they are performed in different planes. On the contrary, the translation of α along α, or $\alpha\times\alpha$, *conveys no conception of a particular plane*; in fact $\alpha\times\alpha$ and $\beta\times\beta$ may be regarded as performed in the same plane. Thus, that which before made the difference does not exist in this case.

Generally, $u.v=u'.v'$, when the two parallelograms generated are equal in magnitude, and lie in the parallel planes; but $u\times u$ and $u'\times u'$ may be always considered as lying in the same plane; and consequently difference of magnitude only remains to constitute a difference between $u\times v$ and $u'\times v'$.

Fig. 28.



* Not *in* but *on* a plane, that is, on one particular side of the plane; *e. g.* on the *upper side* of this sheet of paper.

Again, as regards sign, there are *four* varieties of the form $u.v$, namely $u.v$, $(-u).(-v)$, both $+^{ve}$, and $(-u).v$, $u.(-v)$, both $-^{ve}$. But there are only *two* varieties of the form $u\times v$ when v becomes identical with u; inasmuch as v must be the same as u *in sign as well as in magnitude*, and consequently the notation does not admit of the variations $(-u)\times u$ and $u\times(-u)$. Now the other remaining variations are $u\times u$ and $(-u)\times(-u)$, and these have both the same sign.

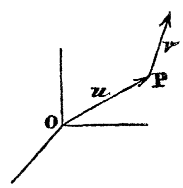
Hence (supposing that u and v are units), since units of lateral translation may differ from each other in two particulars *only*, namely, *sign* and *plane of translation*, and since units of longitudinal translation are incapable of differing in these particulars, we may see the interpretation of the result in art. 37, and its perfect consistency with that in art. 36.

(70.) *Symbol of a line drawn from a given Point.*—If we assume that simple letters, such as u, v, w always denote lines of particular lengths and drawn in particular directions, *but all starting from the Origin of Coordinates* O; then the proper symbol for denoting a line v *drawn from the point* P, OP being u, will be

$$v+u.v+u\times v;$$

for v denotes the line v drawn from O, and $u.v+u\times v$ the lateral and longitudinal effects of translating it from O to P, which effects, as above stated, have reference only to the change of position of P. I shall reserve the consideration of this symbolization for a future occasion, as a striking instance of the same thing will be given in the next section.

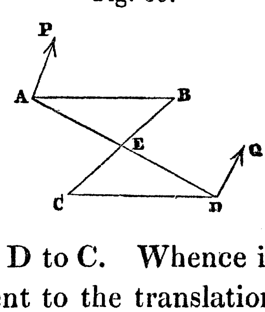
Fig. 29.



II. Statical Application of the Symbolic Forms.

(71.) *Equivalence of Parallel and Equal Translations.*—As a necessary preliminary the *Fundamental Assumption* in art. 29 must be justified. To do this it is only necessary to bear in mind that all statical problems are reducible to the case of balancing forces acting *on the same rigid body*. Now let AB and CD be any two parallel and equal lines in the *same rigid body*; join A and D, C and B; the intersection E bisecting the two joining lines; and let P and Q be two equal parallel forces acting at A and D.

Fig. 30.

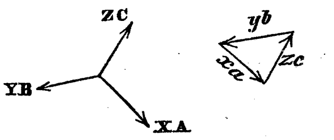


Then P and Q are equivalent to P+Q at E, and P+Q at E is equivalent to P at B and Q at C. We may therefore translate P from A to B, provided we, at the same time, translate Q from D to C. Whence it follows that the translation of Q from C to D must be equivalent to the translation of P from A to B. Thus the fundamental assumption is justified.

(72.) *Representation of Forces by Lines.*—It will be remembered that the general symbolic representation of forces by lines assumes the truth of the Parallelogram of Forces. As a matter of curiosity it may be asked, is it possible to apply this Symbolization of Translation to prove the Parallelogram of forces, *without assuming that* *forces are represented by lines*? It may be done very simply, and I give the proof here as an example of the application of the method.

It may be seen, on referring to art. 21 and the following articles, that there is no assumption whatever of the possibility of representing forces by lines generally. The arrow representing the translated magnitude is used merely as a *conventional symbol*, just as a letter in algebra. I therefore may apply the notation $u.v$ to the present question, provided I consider v to be the symbol of a force, and not of a line representing that force. I must not however employ the reasoning in art. 32, for that distinctly assumes the point in question. The following then is the proof I shall give.

Fig. 31.



(73.) *Parallelogram of Forces.*—Let A, B, C denote *any* three units of force, and a, b, c units of length (directed units) parallel respectively to A, B, C; let X, Y, Z, x, y, z be pure numbers; suppose that the forces XA, YB, ZC balance each other, and that xa, yb, zc are the three sides of a triangle formed by lines drawn parallel to the forces. Then by *successive* addition we have

$$xa+yb+zc=0, \qquad (1.)$$

and by *simultaneous* addition

$$\text{XA}+\text{YB}+\text{ZC}=0. \qquad (2.)$$

Hence, from (1.) and (2.), we have

$$(-zc).(-\text{ZC})=(xa+yb).(\text{XA}+\text{YB});$$

or, observing that (since no lateral effect is produced by translating a magnitude in its own direction) $c.\text{C}$, $a.\text{A}$, $b.\text{B}$ are each zero, we have

$$0=x\text{Y}(a.\text{B})+y\text{X}(b.\text{A}). \qquad (3.)$$

Now, without altering the directions of the units A, B, a, b, let us put X=Y; in which case it is self-evident, that ZC must become equally inclined to XA and YB, and therefore zc must make equal angles with xa and yb, which gives $x=y$. Thus (3.) becomes

$$a.\text{B}+b.\text{A}=0, \text{ or } b.\text{A}=-a.\text{B}.$$

Wherefore, restoring the inequality of X and Y, we find from (3.),

$$(x\text{Y}-y\text{X})a.\text{B}=0, \text{ or } x\text{Y}-y\text{X}=0.$$

And similarly, we may show that

$$y\text{Z}-z\text{Y}=0$$
$$z\text{X}-x\text{Z}=0;$$

whence $\text{X}:\text{Y}:\text{Z}::x:y:z.$

And this is, virtually, the Parallelogram of Forces.

Thus it appears that the notation $u.v$ is capable of affording a simple proof of the great fundamental theorem of Statics; this application of the method is given, however, as I stated above, merely to show by example what can be done in this way. I may observe that the whole of the proof here given depends simply upon *two things*, the distributiveness of $u.v$, and the fact that numerical coefficients may be brought out and incorporated by actual multiplication. I now proceed to give, generally, the mode of applying the notation to determine the conditions of equilibrium of a rigid body, or a system of particles.

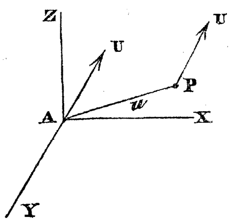
(74.) *Remarkable symbolization of a Force acting at a Specified Point of a Rigid Body.* One of the most remarkable symbolizations which my proposed method leads to appears to me to be the following.

Let U denote a force (in magnitude and direction) *acting at any assumed origin* A, then

$$U+u.U$$

will *completely* denote *the same force supposed to act at the point u*, that is, at the point (P) whose distance from the origin is, symbolically, u. For if we apply the force at the origin A and then translate it to P, it will be the same thing as if we applied the force directly at P. Thus

Fig. 32.



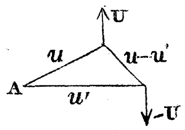
effect of force at P=effect at A (or U)

+effect of translation from A to P.

But, the lateral effect only of the translation need be considered, for the longitudinal effect is zero, inasmuch as we may suppose a force to act at any point of its line of direction on a rigid body. Hence the effect of the force acting at P is symbolically represented by the force U+ lateral effect of translation, or $U+u.U$.

(75.) *Symbol of a Couple.*—Let the couple consist of two forces, U at the point u, and $-U$ at the point u': then these two forces are completely represented, as regards their effect on the rigid body, by the expression

Fig. 33.



$$(U+u.U)+\big(-U+u'.(-U)\big)$$

or

$$(u-u').U,$$

which is the general symbol for a couple; as indeed is clear beforehand from the fact, that the couple is that which translates the force U from the point (u') to the point (u), *i. e.* along the line ($u-u'$).

It will be generally simpler to employ the symbol in the form

$$u.U,$$

u here denoting, symbolically, the line drawn from the point of application of the force ($-U$) to that of the force (U). The *directrix* of this, *i. e.* the *axis of the couple*, is $D(u.U)$.

(76.) *To combine a given Set of Couples.*—Let us put (see art. 15)

$$u=x\alpha+y\beta+z\gamma$$

$$U=X\alpha+Y\beta+Z\gamma.$$

Then we find, by *lateral multiplication*,

$$u.U=(xY-yX)\alpha.\beta+(yZ-zY)\beta.\gamma+(zX-xZ)\gamma.\alpha.$$

Hence, if we suppose the given couples to be $u.U$, $u'.U'$, $u''.U''$, &c., and if we put, for brevity,

$$\Sigma(xY-yX)=N,\ \Sigma(yZ-zY)=L,\ \Sigma(zX-xZ)=M,$$

the combined effect of the couples will be

$$\Sigma(u.\mathrm{U})=\mathrm{L}\beta.\gamma+\mathrm{M}\gamma.\alpha+\mathrm{N}\alpha.\beta$$
$$=\mathrm{D}^{-1}(\mathrm{L}\alpha+\mathrm{M}\beta+\mathrm{N}\gamma) \text{ (art. 48)}.$$

Hence, if we assume G to denote the magnitude and γ' the direction of $\mathrm{L}\alpha+\mathrm{M}\beta+\mathrm{N}\gamma$, which gives (art. 17)

$$\mathrm{G}^2=\mathrm{L}^2+\mathrm{M}^2+\mathrm{N}^2 \ldots\ldots (1.), \qquad \gamma'=\frac{\mathrm{L}}{\mathrm{G}}\alpha+\frac{\mathrm{M}}{\mathrm{G}}\beta+\frac{\mathrm{N}}{\mathrm{G}}\gamma \ldots\ldots (2.);$$

we find

$$\Sigma u.\mathrm{U}=\mathrm{D}^{-1}(\mathrm{G}\gamma')$$
$$=\mathrm{G}\alpha'.\beta' \text{ (arts. 48 and 16)}.$$

Here $\alpha'.\beta'$ denotes a unit-couple in a plane at right angles to γ'; and thus the resultant is a couple G in this plane; G and γ' being given by (1.) and (2.).

(77.) *To combine a given Set of Forces acting at given Points of a Rigid Body.*—Let the forces be U, U′, U,″ &c., and their points of application u, u', u'', &c.; then, by art. 74, their combined effect will be

$$\Sigma(\mathrm{U}+u.\mathrm{U}).$$

Hence, if we put

$$\Sigma\mathrm{X}=\mathrm{X}_{,}, \quad \Sigma\mathrm{Y}=\mathrm{Y}_{,}, \quad \Sigma\mathrm{Z}=\mathrm{Z}_{,},$$

and employ the notation of the preceding article, the combined effect becomes

$$\mathrm{X}_{,}\alpha+\mathrm{Y}_{,}\beta+\mathrm{Z}_{,}\gamma+\mathrm{G}\alpha'.\beta',$$

or

$$\mathrm{R}_{,}\alpha_{,}+\mathrm{G}\alpha'.\beta',$$

where

$$\mathrm{R}_{,}^2=\mathrm{X}_{,}^2+\mathrm{Y}_{,}^2+\mathrm{Z}_{,}^2, \ldots\ldots\ldots\ldots (3.)$$

and

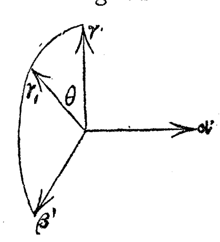
$$\gamma_{,}=\frac{\mathrm{X}_{,}}{\mathrm{R}_{,}}\alpha+\frac{\mathrm{Y}_{,}}{\mathrm{R}_{,}}\beta+\frac{\mathrm{Z}_{,}}{\mathrm{R}_{,}}\gamma. \ldots\ldots\ldots\ldots (4.)$$

Thus it appears that the set of given forces are combined into a single force $\mathrm{R}_{,}\alpha_{,}$ (given by (3.) and (4.)), and a single couple $\mathrm{G}\alpha'.\beta'$ (given by (1.) and (2.) previous article).

From the result just obtained the various well-known conditions and equations, relating to the effect of a set of forces on a rigid body, immediately follow. To exemplify the method I shall apply the formula $\mathrm{R}_{,}\alpha_{,}+\mathrm{G}\alpha'.\beta'$ to the following question.

(78.) *To combine the set of forces into Two Forces.*—Let us so choose α' and β' (which are arbitrary, except so far as they are perpendicular to γ'), that β' shall be in the same plane as $\gamma_{,}$ and γ'; and let θ be the angle which $\gamma_{,}$ makes with γ'. Then, by article 17,

Fig. 34.



$$\gamma_{,}=\gamma'\cos\theta+\beta'\sin\theta;$$
$$\therefore\ \mathrm{R}_{,}\gamma_{,}+\mathrm{G}\alpha'.\beta'=(\mathrm{R}_{,}\cos\theta)\gamma'+(\mathrm{R}_{,}\sin\theta)\beta'+\mathrm{G}\alpha'.\beta'$$
$$=(\mathrm{R}_{,}\cos\theta)\gamma'+(\mathrm{R}_{,}\sin\theta)\beta'+\left(\frac{\mathrm{G}}{\mathrm{R}_{,}\sin\theta}\alpha'\right)\cdot(\mathrm{R}_{,}\sin\theta)\beta'.$$

Now by article 74 this is the expression for the effect of two forces, viz.—

$$\mathrm{R}_{,}\cos\theta \text{ in the direction } \gamma',$$

and

$$\mathrm{R}_{,}\sin\theta \text{ in the direction } \beta',$$

the former acting at the origin, and the latter at a point whose distance from the origin is

$$\frac{G}{R_{\prime} \sin \theta} \text{ drawn in the direction } \alpha'.$$

$R_{\prime}$, G, γ' and $\gamma_{\prime}$ are given by the equations (1), (2), (3), (4) above; α' is determined because it is at right angles to the plane $(\gamma'\gamma_{\prime})$, and β', because it is in the same plane, and at right angles to γ'. As regards θ, we have, taking the *longitudinal product* of (2.) and (4.),

$$\frac{LX_{\prime}+MY_{\prime}+NZ_{\prime}}{LR_{\prime}}=\gamma'\times\gamma_{\prime}=\cos\theta.$$

(79.) *Centre of Parallel Forces.*—Let the magnitudes of the parallel forces be R, R', R'', &c., $\gamma_{\prime}$ their common direction, and u, u', u'', &c. their points of application. Then their combined effect is

$$\Sigma(R\gamma_{\prime})+\Sigma(u\,.\,R\gamma_{\prime})=(\Sigma R)\gamma_{\prime}+\frac{\Sigma Ru}{\Sigma R}\cdot(\Sigma R)\gamma_{\prime}.$$

Now, by art. 74, this is the symbol for a force ΣR, acting in the direction $\gamma_{\prime}$, and at a point whose distance from the origin is

$$\frac{\Sigma Ru}{\Sigma R},$$

or

$$\frac{\Sigma Rx}{\Sigma R}\alpha+\frac{\Sigma Ry}{\Sigma R}\beta+\frac{\Sigma Rz}{\Sigma R}\gamma,$$

which expresses the common formulæ for the centre of parallel forces.

III. Application of the Symbolic Forms to Dynamics.

(80.) *Effective Force, Vis-Viva, Work.*—If u be the distance of a moving particle m from the origin at any time t, it is clear that du represents, in magnitude and direction, the space described in the time dt; and thus the complete symbol of the velocity becomes

$$\frac{du}{dt}.$$

Also $d\left(\frac{du}{dt}\right)$ represents, in magnitude and direction, the alteration of velocity in the time dt; and thus the complete symbol of the effective force is

$$m\frac{d^2u}{dt^2}.$$

Again, if U denote any force acting on m, it is easy to see that the symbol of the work accumulated, while m is moving from the point u to the point u', is

$$\int_u^{u'} U\times du.$$

Lastly, if for U here we put the effective force, the *effective work* will be

$$\int_u^{u'} m\frac{d^2u}{dt^2}\times du$$

or* $$\frac{m}{2}\left(\frac{du'}{dt}\times\frac{du'}{dt}-\frac{du}{dt}\times\frac{du}{dt}\right).$$

Here $\frac{du}{dt}\times\frac{du}{dt}$ denotes the square of the velocity (art. 49), and thus the expression denotes the ordinary *vis-viva.*

(81.) *Description of Areas.*—It is clear that $\frac{1}{2}u.du$ is the area described in the time dt by u; for it is the half parallelogram formed on u and du. But it is to be noticed that $\frac{1}{2}u.du$ represents this area, not only in magnitude, but also *in position.*

If we put $\frac{1}{2}u.du=\text{A}dt$, we find

$$\frac{d\text{A}}{dt}=\frac{1}{2}\left(u.\frac{d^2u}{dt^2}+\frac{du}{dt}\cdot\frac{du}{dt}\right), \text{ and } \frac{du}{dt}\cdot\frac{du}{dt}=0 \text{ (art. 32)};$$

whence $$\frac{d\text{A}}{dt}=\frac{1}{2}u.\frac{d^2u}{dt^2}. \qquad (1.)$$

A, here, is a symbol which represents two important things:—1st, *in magnitude*, it is the ordinary *rate of description of area* $\left(r^2\frac{d\theta}{dt}\right)$; 2ndly, its plane is the plane containing the radius vector and the direction of motion, *i. e.* the *plane of the orbit* of m.

DA is the symbol of a line perpendicular to the plane of the orbit, and equal in magnitude to the rate of description of area.

If we put $u=x\alpha+y\beta+z\gamma$, and perform the operation indicated in equation (1.), we find,

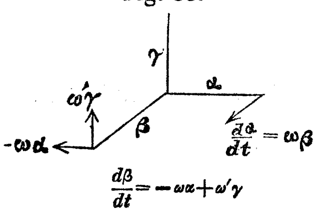
$$2\frac{d(\text{DA})}{dt}=\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)\gamma+\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right)\alpha+\left(z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}\right)\beta.$$

(82.) *Expression for Effective Force with reference to radius vector, angular velocity, and plane of orbit.*—Let r denote the magnitude and α the "*direction*" of u; let β denote a direction *in* the plane of the orbit of m (and, of course, at right angles to α); then γ will be a direction always perpendicular to the plane of the orbit: lastly, let ω denote the angular velocity of u, and ω' the angular velocity of the plane of the orbit about u, ω and ω' being numerical quantities. On referring to art. 19, the following relations are manifest,

$$u=r\alpha \quad \frac{d\alpha}{dt}=\omega\beta \quad \frac{d\beta}{dt}=-\omega\alpha+\omega'\gamma;$$

for $\frac{d\alpha}{dt}$ and $\frac{d\beta}{dt}$ denote, in magnitude and direction, the velocities of the extremities of the directed units α and β, and the figure will show what their values are in terms of ω and ω'.

Fig. 35.



* $d\left(\frac{du}{dt}\times\frac{du}{dt}\right)=\frac{d^2u}{dt}\times\frac{du}{dt}+\frac{du}{dt}\times\frac{d^2u}{dt}$
$=2\frac{d^2u}{dt^2}\times du$ (art. 33).

Hence we immediately find the required expression for the effective force $\frac{d^2u}{dt^2}$ by simple differentiation, as follows,

$$\frac{du}{dt}=\frac{dr}{dt}\alpha+r\frac{d\alpha}{dt}=\frac{dr}{dt}\alpha+r\omega\beta$$

$$\frac{d^2u}{dt^2}=\frac{d^2r}{dt^2}\alpha+\frac{dr}{dt}\frac{d\alpha}{dt}+\frac{d(r\omega)}{dt}\beta+r\omega\frac{d\beta}{dt},$$

or effective force $$=\left(\frac{d^2u}{dt^2}-r\omega^2\right)\alpha+\left(\frac{dr}{dt}\omega+\frac{d(r\omega)}{dt}\right)\beta+r\omega\omega'\gamma.$$

Hence it appears that the effective force is equivalent to the well-known expressions along and perpendicular to the radius vector, together with a third part, $r\omega\omega'$ perpendicular to the plane of the orbit.

(83.) *The use of the forms* u.v *and* u×v *exemplified in the case of motion about a centre of force varying as* r^{-2}.—This case appears to me to afford so good an illustration of the use of these forms, that I shall give it here briefly. Assuming α, β, γ, ω, r as in the preceding article, it is clear that the symbol of the central force is $-\frac{\mu}{r^2}\alpha$, and therefore we have

$$\frac{d^2u}{dt^2}=-\frac{\mu}{r^2}\alpha, \quad \ldots \ldots \ldots \quad (1.)$$

whence $$u.\frac{d^2u}{dt^2}=0, \text{ since } u=r\alpha, \text{ and } \alpha.\alpha=0,$$

therefore (by art. 81), $$\frac{dA}{dt}=0. \quad \ldots \ldots \ldots \quad (2.)$$

This indicates that there is no variation of A, and consequently (see art. 81) that the plane of the orbit, and the rate of description of area is constant.

If we put for A its value (art. 81) $\frac{1}{2}u.\frac{du}{dt}$ and for u and $\frac{du}{dt}$ their values $r\alpha$ and $\frac{d(r\alpha)}{dt}$, observing that $\frac{d\alpha}{dt}=\omega\beta$, we find

$$A=\frac{1}{2}r\alpha.\left(\frac{dr}{dt}\alpha+r\omega\beta\right)=\frac{1}{2}r^2\omega\alpha.\beta,$$

or, by (2.), $$A=\frac{1}{2}h\alpha.\beta \quad (h=r^2\omega, \text{ as usual}).$$

Now it is singular that (1.) admits of *immediate integration*, instead of requiring the well-known transformations: for, in it, put for α its value, $-\frac{1}{\omega}\frac{d\beta}{dt}$ (see art. 82), and we have

$$\frac{d^2u}{dt^2}=\frac{\mu}{r^2\omega}\frac{d\beta}{dt}=\frac{\mu}{h}\frac{d\beta}{dt};$$

wherefore, integrating, we find

$$\frac{du}{dt}=\frac{\mu}{h}\beta+\text{ constant}.$$

Here put for $\frac{du}{dt}$ its value, and there results

$$\frac{dr}{dt}\alpha+r\omega\beta=\frac{\mu}{h}\beta+\text{ constant } (c\beta' \text{ suppose*}).$$

Multiply this *longitudinally* by β, and, since $\beta\times\alpha=0$, $\beta\times\beta=1$, and $\beta\times\beta'=\alpha\times\alpha'$ $=\cos\theta$, we find

$$r\omega=\frac{\mu}{h}+c\cos\theta;$$

or, since $r\omega=\frac{h}{r}$, this gives

$$\frac{1}{r}=\frac{\mu}{h^2}+\frac{c}{h}\cos\theta,$$

the well-known equation, θ being the angle which α (the direction of the radius vector) makes with α' (an arbitrary constant direction).

The manner in which the symbolic forms have effected this integration appears to me to be worthy of notice.

(84.) *General Expression for the Momentum of a Rigid Body moving in any manner.*—The *Momentum* of a particle (m), moving with a velocity $\left(\frac{du}{dt}\right)$, is

$$m\frac{du}{dt}.$$

And this symbol represents the Momentum in direction as well as magnitude. Now momentum is really *a force*, estimated, however, somewhat differently from ordinary pressure, on the principle that the true *dynamical effect* of a force is proportional to absolute intensity and the time of its action conjointly†. Hence, regarding the momentum of m as a force, its complete symbol will be (by art. 74),

$$\frac{mdu}{dt}+u.\left(\frac{mdu}{dt}\right).$$

Thus the symbol of the *total momentum* of the rigid body will be

$$\Sigma m\left\{\frac{du}{dt}+u.\frac{du}{dt}\right\}. \quad \quad (1.)$$

(85.) If $\bar{u}$ denote the distance of the *centre of gravity* from the origin, and u' the distance of the point (u) from the centre of gravity, we have $u=\bar{u}+u'$, and thus, since $\Sigma mu'=0$, (1.) becomes (putting M for Σm)

$$M\frac{d\bar{u}}{dt}+\bar{u}.\left(M\frac{d\bar{u}}{dt}\right)+\Sigma m\left(u'.\frac{du'}{dt}\right). \quad \quad (2.)$$

* Of course the constant is the symbol for some constant line, as regards *direction* as well as magnitude, and therefore I put it in the form $c\beta'$, c being a number and β' a "*direction.*"

† If the *pressure* P produces a velocity v in the time t, it produces the velocity $\frac{v}{t}$ per second, and therefore $P=m\frac{v}{t}$; or $mv=Pt$. Pt then is the momentum, and this is proportional to P and t conjointly.

Now here, by (1.), $M\frac{d\bar{u}}{dt}+\bar{u}.\left(M\frac{d\bar{u}}{dt}\right)$ is the expression for the momentum of M concentrated into the centre of gravity, and $\Sigma m\left(u'.\frac{du'}{dt}\right)$ is the momentum of the body as regards its motion *relatively* to the centre of gravity (as is evident from (1.), for, if the origin have no momentum, $\Sigma m\frac{du}{dt}=0$).

In Section V. I shall show, that

$$\Sigma m\left(u'.\frac{du'}{dt}\right)=D^{-1}(A\omega_1\alpha+B\omega_2\beta+C\omega_3\gamma),$$

that is, *a couple whose axis* is $A\omega_1\alpha+B\omega_2\beta+C\omega_3\gamma$. Here A, B, C are the moments of inertia about the three principal axes, of which α, β, γ are supposed to be the directions; and ω_1, ω_2, ω_3 are the well-known component angular velocities.

(86.) *General Expression for the Energy of a Rigid Body moving in any manner.*— I venture to suggest the word "*energy*" as a proper designation for the *total effective force* by which the motion of a rigid body is produced, inasmuch as ἐνέργεια means force *actually exerted and effective.* The symbol, then, of *the energy* of a rigid body will be

$$\Sigma m\left(\frac{d^2u}{dt^2}+u.\frac{d^2u}{dt^2}\right).$$

Now, observing that $\frac{du}{dt}.\frac{du}{dt}=0$, we have

$$u.\frac{d^2u}{dt^2}=d\left(u.\frac{du}{dt}\right).$$

Hence the expression for the energy is

$$\frac{d}{dt}\left\{\Sigma m\left(\frac{du}{dt}+u.\frac{du}{dt}\right)\right\}.$$

Comparing this with the expression for the *momentum* in art. 84, we have

$$energy=\frac{d\,(momentum)}{dt} \quad \ldots\ldots\ldots\ldots \quad (1.)$$

(87.) This, though a very simple result, is really one of importance; thus if the centre of gravity be fixed, we find (by art. 85.)

$$energy=D^{-1}\frac{d}{dt}(A\omega_1\alpha+B\omega_2\beta+C\omega_3\gamma).$$

Now it will be shown in Sect. V. that

$$\frac{d\alpha}{dt}=D\omega.\alpha,\ \frac{d\beta}{dt}=D\omega.\beta,\ \frac{d\gamma}{dt}=D\omega.\gamma,$$

where $\qquad \omega=\omega_1\alpha+\omega_2\beta+\omega_3\gamma.$

Hence, performing the differentiation, and observing that

$$D^{-1}\alpha=\beta.\gamma,\ D^{-1}\beta=\gamma.\alpha,\ D^{-1}\gamma=\alpha.\beta,\ \text{and}\ D^{-1}D=1,$$

we find $$energy = A\frac{d\omega_1}{dt}\beta.\gamma + B\frac{d\omega_2}{dt}\gamma.\alpha + C\frac{d\omega_3}{dt}\alpha.\beta$$
$$+\omega.(A\omega_1\alpha + B\omega_2\beta + C\omega_3\gamma).$$

Lastly, if we perform the *lateral multiplication* denoted by $\omega.$ here, we find

$$energy = \left\{A\frac{d\omega_1}{dt} + (C-B)\omega_2\omega_3\right\}\beta.\gamma$$
$$+\left\{B\frac{d\omega_2}{dt} + (A-C)\omega_3\omega_1\right\}\gamma.\alpha$$
$$+\left\{C\frac{d\omega_3}{dt} + (B-A)\omega_1\omega_2\right\}\alpha.\beta.$$

I need not point out the meaning of this formula in relation to EULER's equations for a rigid body.

(88.) *General Expression for the "Vis-Mortua" or "Dead Pull" on a Rigid Body.*—I may use the almost obsolete word "*Vis-Mortua*" (which has been so well translated by the familiar expression "*Dead Pull*") in the sense in which it was originally employed to denote simple pressure or *impressed force*. I shall therefore designate the total mechanical effect of the impressed forces acting on a rigid body as the *Vis-Mortua* or *Dead Pull* on that body. If U denote the force acting at the point (u), we have, therefore, by art. 74,

$$Vis\text{-}Mortua = \Sigma(U + u.U):$$

this of course is the same expression as that in art. 77.

If we put, as before, $u = \bar{u} + u'$, this expression becomes

$$\Sigma U + \bar{u}.\Sigma U + \Sigma u'.U.$$

Here $\Sigma U + \bar{u}.\Sigma U$ is the symbol of the force ΣU acting at the point ($\bar{u}$), and $\Sigma u'.U$ is (art. 75) the symbol of a couple.

(89.) As an example, the result of which I shall require in Sect. V., I shall calculate the Vis-Mortua of a rigid body acted on by the attraction of a distant particle m', taking the centre of gravity of the rigid body as origin, and α, β, γ as the directions of the three principal axes; assuming also u' to denote the distance of m', r' the magnitude of u', and r that of u.

Here U is a force acting along the line joining the two points (u) and (u'), and inversely proportional to the square of the magnitude of that line. The line alluded to is $u'-u$, its magnitude is $\sqrt{(u'-u)\times(u'-u)}$, and its direction therefore is

$$\frac{u'-u}{\sqrt{(u'-u)\times(u'-u)}}.$$

Hence $$U = mm'\frac{(u'-u)}{\{(u'-u)\times(u'-u)\}^{\frac{3}{2}}}.$$

Also $$\{(u'-u)\times(u'-u)\}^{-\frac{3}{2}} = (u'\times u' - 2u'\times u + u\times u)^{-\frac{3}{2}}$$
$$= (r'^2 - 2u'\times u + r^2)^{-\frac{3}{2}}$$
$$= \frac{1}{r'^3}\left(1 + \frac{3u'\times u}{r'^2}\right) \text{ nearly.}$$

Hence, observing that $u.u=0$, and $\Sigma mu=0$, we have

$$\Sigma u.\text{U}=\frac{3m'}{r'^5}\Sigma m(u'\times u)(u.u')$$
$$=-\frac{3m'}{r'^5}u'.\{\Sigma m(u'\times u)u\}.$$

Now $\Sigma m(u'\times u)u=\Sigma m(xx'+yy'+zz')(x\alpha+y\beta+z\gamma)$

$$=(\Sigma mx^2)x'\alpha+(\Sigma my^2)y'\beta+(\Sigma mz^2)z'\gamma, \text{ by properties of principal axes,}$$
$$=(\Sigma mr^2-\text{A})x'\alpha+(\Sigma mr^2-\text{B})y'\beta+(\Sigma mr^2-\text{C})z'\gamma$$
$$=(\Sigma mr^2)u'-(\text{A}x'\alpha+\text{B}y'\beta+\text{C}z'\gamma).$$

Wherefore, observing that $u'.u'=0$, we find

$$\Sigma u.\text{U}=\frac{3m'}{r'^5}u'.(\text{A}x'\alpha+\text{B}y'\beta+\text{C}z'\gamma).$$

Also $\qquad \Sigma\text{U}=\Sigma mm'\,\frac{u'-u}{r'^3}$ (neglecting $\frac{3u'\times u}{r'^2}$ for obvious reasons)

$$=\text{M}m'\frac{u'}{r'^3} \qquad \begin{cases}\text{M}=\Sigma m\\ \Sigma mu=0.\end{cases}$$

Hence we have

$$\textit{Vis-Mortua}=\text{M}m'\,\frac{u'}{r'^3}+\frac{3m'}{r'^5}u'.(\text{A}x'\alpha+\text{B}y'\beta+\text{C}z'\gamma),$$

which expresses a force $\text{M}m'\frac{u'}{r'^3}$ acting at the origin, and a couple,

$$\frac{3m'}{r'^5}u'.(\text{A}x'\alpha+\text{B}y'\beta+\text{C}z'\gamma).$$

If we perform the *lateral multiplication* indicated by $(u'.)$, this couple becomes

$$\frac{3m'}{r'^5}\Big\{(\text{C}-\text{B})y'z'\beta.\gamma+(\text{A}-\text{C})z'x'\gamma.\alpha+(\text{B}-\text{A})x'y'\alpha.\beta\Big\},$$

which gives the three well-known couples in the theory of Precession and Nutation.

IV. Application of the Symbolic Forms to determine the Correction for the Earth's Rotation in Problems relating to motion on or near the Earth's Surface.

(90.) The best way of defining that which is commonly called the *Centrifugal Force* appears to be the following, viz. that it is an imaginary force which may be introduced as a correction for the error of not taking into account the rotation of the radius vector r. Suppose P to denote the accelerating force acting along r, and let us for a moment forget that r has an angular velocity $\left(\frac{d\theta}{dt}\right)$, then we put

$$\frac{d^2r}{dt^2}=\text{P};$$

but this is erroneous, and we must correct it for the rotation by adding to P the term $r\left(\frac{d\theta}{dt}\right)^2$, as is well known. Hence we may regard the centrifugal force as a *correction for neglected rotation.*

But it only corrects the error so far as the motion along r is concerned; another correction (supposing still the rotation is neglected or forgotten) is necessary to be applied at right angles to r, namely, the imaginary force $-\frac{1}{r}\frac{d}{dt}\left(\frac{r^2d\theta}{dt}\right)$. Thus the true and complete correcting force is the resultant of the two forces

$$r^2\frac{d\theta}{dt}, \text{ and } -\frac{1}{r}\frac{d}{dt}\left(\frac{r^2d\theta}{dt}\right).$$

It appears to me that the idea here suggested might be applied with great advantage to cases of motion on or near the earth's surface. The beautiful pendulum experiment which made so much noise last year, and the various investigations respecting it, give great interest to such cases of motion. I propose therefore to investigate here, by the aid of the Symbolic Forms, the proper symbol of the imaginary force which corrects completely for the earth's rotation supposed to be neglected. By the aid of this symbol, it will be found that the greatest possible simplicity is introduced into investigations such as those relating to the pendulum experiment. It will enable the investigator to forget altogether the earth's rotation in framing his equations of motion, and at the same time to *correct his error by the introduction of a simple term.*

(91.) Let ω denote a line pointing in the direction of the earth's polar axis (north suppose), and representing, by its length, the earth's angular velocity. In other words, let ω be the *directrix of the earth's rotation*, then (as will be shown in Sect. V.) it is easy to see, that, if u denote (symbolically) the distance of any point from the earth's centre, the velocity communicated to it by the earth's rotation (if it be fixed to the earth) is represented by the symbol*

$$\mathrm{D}\omega.u.$$

Now let d denote differentiation (of u) on the erroneous supposition that the earth is fixed, and δ the true and complete differentiation; then the true velocity of the point u is $\frac{\delta u}{dt}$, and this must be the resultant of the erroneous velocity $\left(\frac{du}{dt}\right)$ and the velocity ($\mathrm{D}\omega.u$) due to the rotation. Hence we have

$$\frac{\delta u}{dt}=\frac{du}{dt}+\mathrm{D}\omega.u.$$

The effective accelerating force will be obtained by the true and complete differentiation of the correct velocity, that is,

$$\frac{\delta}{dt}\left(\frac{du}{dt}+\mathrm{D}\omega.u\right),$$

or

$$\frac{d}{dt}\left(\frac{\delta u}{dt}\right)+\mathrm{D}\omega.\frac{\delta u}{dt} \quad \text{(observing that } \omega \text{ is a constant),}$$

* For $\mathrm{D}\omega.u$ denotes a line at right angles to both ω and u, and its magnitude is $nr\sin\theta$; where n is the magnitude of ω, r that of u, and θ the angle which u makes with ω. Therefore $\mathrm{D}\omega.u$ is manifestly the velocity caused in the point u by the rotation ω.

which, putting for $\frac{\delta u}{dt}$ the above value, becomes

$$\frac{d^2u}{dt^2}+2D\omega.\frac{du}{dt}+D\omega.(D\omega.u).$$

Hence if U denote (symbolically) the resultant of the accelerating forces, whatever they may be, which act on the point (u), we find

$$\frac{d^2u}{dt^2}+2D\omega.\frac{du}{dt}+(D\omega.)^2u=U. \quad \text{(1.)}$$

I may observe, in passing, that, since $d+(D\omega.)$ represents the *complete* differentiation of u, we might have written down the equation of motion *immediately*, in the form

$$\frac{\{d+(D\omega.)\}^2}{dt^2}u=U,$$

which, expanded, is identical with (1.).

Now, if we had forgotten the earth's rotation, we should have put, instead of (1.),

$$\frac{d^2u}{dt^2}=U.$$

Hence it appears that, if we neglect the rotation in forming the equation of motion, we may *correct the error*, by supposing that there is the imaginary force

$$-\left\{2D\omega.\frac{du}{dt}+(D\omega.)^2u\right\} \quad \text{(2.)}$$

acting in addition to the real force represented by U; for on this supposition we find

$$\frac{d^2u}{dt^2}=U-2D\omega.\frac{du}{dt}-(D\omega.)^2u, \quad \text{(3.)}$$

which is equivalent to (1.).

As regards terrestrial problems, however, the expression (2.) admits of an important simplification; for the accelerating force of gravity (g), which of course is included in U, is supposed to be the resultant of the earth's attraction, and the *common centrifugal force*. Now this common centrifugal force is that which is conceived to be in action upon a particle rigidly connected with the revolving earth; in other words, it is what (2.) becomes when $\frac{du}{dt}=0$. Wherefore the expression for the *common centrifugal force* is

$$-(D\omega.)^2u. \quad \text{(4.)}$$

As this therefore is included among the forces which U represents, it ought to be omitted in the equation (3.). Thus we find that

$$-2D\omega.\frac{du}{dt} \quad \text{(5.)}$$

is the force which must be supposed to act on the point (u) *as a correction for the neglected rotation.*

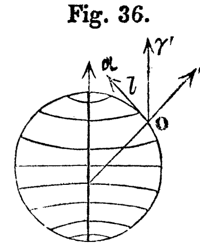
We may, therefore, in all problems of motion relative to the earth, forget altogether the earth's rotation, provided we introduce the force (5.), in addition to the accelerating forces, whatever they may be, which are really in action upon the point (u); it being understood that the common centrifugal force is allowed for in g.

(92.) In (5.) $\frac{du}{dt}$ is the apparent velocity of the point u (apparent, that is, to an observer unconscious of the earth's motion); and if n denote the earth's angular velocity, and γ' the direction of the polar axis, $\omega = n\gamma'$. Thus (5.) becomes

$$-2n\mathrm{D}\gamma'.\frac{du}{dt}. \qquad (6.)$$

Now this represents a force at right angles to γ' and $\frac{du}{dt}$, *i. e.* to the *polar plane* in which the apparent velocity is taking place at the instant t. Also the magnitude of this force is $2n$ times the apparent velocity $\left(\frac{du}{dt}\right)$ multiplied by the sine of the angle it $\left(\frac{du}{dt}\right)$ makes with the polar axis (γ').

Fig. 36.



(93.) This force may be expressed with reference to horizontal and vertical coordinates at any place, as follows:

Let O be the place, γ the vertical at O, and then the plane ($\alpha\beta$) will be horizontal: also let α be chosen so as to lie in the meridian plane; and let l denote the latitude of O (*i. e.* the angle γ' makes with α). Then

$$\gamma' = \alpha \cos l + \gamma \sin l. \qquad (7.)$$

Also if we take u to denote the distance of the moving point from O at any time t, and therefore put

$$u = x\alpha + y\beta + z\gamma, \qquad (8.)$$

$\frac{du}{dt}$ will be the same as the $\frac{du}{dt}$* in (6.), for all that we have to express by $\frac{du}{dt}$ is the apparent velocity of the point u.

Hence, differentiating (8.), and performing the operations indicated by $2n\mathrm{D}\gamma'.$, (6.) becomes

$$-2n\mathrm{D}(\alpha \cos l + \gamma \sin l).\left(\frac{dx}{dt}\alpha + \frac{dy}{dt}\beta + \frac{dz}{dt}\gamma\right),$$

or

$$2n\left\{\left(\frac{dy}{dt}\sin l\right)\alpha + \left(-\frac{dx}{dt}\sin l + \frac{dz}{dt}\cos l\right)\beta - \left(\frac{dy}{dt}\cos l\right)\gamma\right\}. \qquad (9.)$$

Hence the ordinary equations of motion will be

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \mathrm{X} + 2n\frac{dy}{dt}\sin l \\ \frac{d^2y}{dt^2} &= \mathrm{Y} - 2n\frac{dx}{dt}\sin l + 2n\frac{dz}{dt}\cos l \\ \frac{d^2z}{dt^2} &- \mathrm{Z} - 2n\frac{dy}{dt}\cos l \end{aligned}\right\} \qquad (10.)$$

* Originally u was measured from the earth's centre.

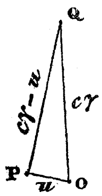
where X, Y, Z represent the accelerating forces, whatever they may be, that are in action on the point u.

(94.) The formation of these equations, here given, affords a good example of the use of the symbolic form $u.v$: but, to illustrate the method more clearly, it will be worth while to employ, in some particular problem, the general equation,

$$\frac{d^2u}{dt^2}=\mathrm{U}-2n\mathrm{D}\gamma'.\frac{du}{dt}, \quad \ldots \ldots \ldots \quad (11.)$$

which includes the three equations (10.), U denoting $\mathrm{X}\alpha+\mathrm{Y}\beta+\mathrm{Z}\gamma$. The problem I shall choose will be that of the *Pendulum Experiment.*

Fig. 37.



Let QP represent the string at any time t, QO its vertical position, c its length; then

$$\mathrm{OQ}=c\gamma, \quad \mathrm{OP}=u, \quad \mathrm{PQ}=c\gamma-u,$$

also *direction* of $\mathrm{PQ}=\frac{c\gamma-u}{c}$.

Let T denote (in magnitude) the tension of PQ; and then, since the direction of T is that of PQ, the symbol of the tension is $\mathrm{T}\,\frac{c\gamma-u}{c}$; to which if we add $-g\gamma$, the symbol of the force of gravity, we find

$$\mathrm{U}=\mathrm{T}\,\frac{c\gamma-u}{c}-g\gamma=(\mathrm{T}-g)\gamma-\frac{\mathrm{T}}{c}u.$$

Hence (11.) becomes

$$\frac{d^2u}{dt^2}=(\mathrm{T}-g)\gamma-\frac{\mathrm{T}}{c}u-2n\mathrm{D}\gamma'.\frac{du}{dt}. \quad \ldots \ldots \ldots \quad (12.)$$

Now, for greater simplicity, I shall suppose that u represents a small excursion, and c a long string. On this supposition we may regard u as always horizontal. Also, if we put for γ' its value (7.), the equation (12.) becomes

$$\left\{\frac{d^2u}{dt^2}+2n\sin l\mathrm{D}\gamma.\frac{du}{dt}-\frac{\mathrm{T}}{c}u\right\}+\left\{2n\cos l\mathrm{D}\alpha.\frac{du}{dt}-(\mathrm{T}-g)\gamma\right\}=0.$$

Hence, since $\frac{du}{dt}$ is horizontal, $\mathrm{D}\alpha.\frac{du}{dt}$ is vertical, and $\mathrm{D}\gamma.\frac{du}{dt}$ is horizontal. The equation (12.), therefore, is separated into two parts, horizontal and vertical, which, being equated to zero, we obtain

$$\frac{d^2u}{dt^2}+2n\sin l\mathrm{D}\gamma.\frac{du}{dt}-\frac{\mathrm{T}}{c}u=0 \quad \ldots \ldots \ldots \quad (13.)$$

$$(\mathrm{T}-g)\gamma=2n\cos l\mathrm{D}\alpha.\frac{du}{dt}. \quad \ldots \ldots \ldots \quad (14.)$$

We have to substitute for T in (13.) its value derived from (14.), which on account of the smallness of n, and the fact that T is multiplied by $\frac{u}{c}$ in (13.), gives $\mathrm{T}=g$ for a first approximation. Hence (13.) becomes

$$\frac{d^2u}{dt^2}+2n\sin l\mathrm{D}\gamma.\frac{du}{dt}-\frac{g}{c}u=0. \quad \ldots \ldots \ldots \quad (15.)$$

Now, here, the operation (Dγ.) is performed only on lines at right angles to γ; we may therefore put $\sqrt{-1}$ for (Dγ.) (see art. 55); and thus (15.) becomes

$$\frac{d^2u}{dt^2}+2n\sin l\sqrt{-1}\frac{du}{dt}-\frac{g}{c}u=0. \quad \text{(16.)}$$

The roots of the equation

$$\left(\frac{d}{dt}\right)^2+2n\sin l\sqrt{-1}\left(\frac{d}{dt}\right)-\frac{g}{c}=0$$

are

$$-n\sin l\sqrt{-1}\pm\sqrt{-n^2\sin^2 l-\frac{g}{c}},$$

or

$$(-n\sin l\pm m)\sqrt{-1},$$

if, for brevity, we put $n^2\sin^2 l+\frac{g}{c}=m^2$. Hence the solution of (16.) is

$$u=\varepsilon^{-nt\sin l\sqrt{-1}}(\text{A}\varepsilon^{mt\sqrt{-1}}+\text{B}\varepsilon^{-mt\sqrt{-1}}). \quad \text{(17.)}$$

The constants A and B here denote two arbitrary lines in the horizontal plane.

If the earth were fixed, the form of the solution would have been

$$u=\text{A}\varepsilon^{mt\sqrt{-1}}+\text{B}\varepsilon^{-mt\sqrt{-1}} \qquad \left\{\text{for then } m^2=\frac{g}{c}\right\}.$$

Now $\varepsilon^{-nt\sin l\sqrt{-1}}$* indicates a uniform *backward* rotation of $\text{A}\varepsilon^{mt\sqrt{-1}}+\text{B}\varepsilon^{-mt\sqrt{-1}}$ with an angular velocity $n\sin l$. Thus it appears that the apparent curve of motion of the pendulum will be the same form as if the earth were fixed, only there will be a slow angular regression of the whole about the vertical as axis at the rate $n\sin l$ per second.

I may observe, in passing, that the simplest interpretation of (17.) is this; that the motion of the point (u) results from the superposition of two motions,

$$\text{A}\varepsilon^{(m-n\sin l)t\sqrt{-1}} \text{ and } \text{B}\varepsilon^{-(m+n\sin l)t\sqrt{-1}};$$

and these are two uniform circular* motions, the former that of the line A forward with an angular velocity $(m-n\sin l)$; the latter that of B backward with an angular velocity $m-n\sin l$.

(95.) As my object is simply to exemplify the application of my notation, I shall not proceed to a second approximation; which however is very easily effected by substituting for T in (13.) its complete value given by (14.), after having put for u in (14.) the value (17.) just obtained. The result is important, especially as regards motion near the equator.

* $\text{A}\varepsilon^{\theta\sqrt{-1}}$ denotes A turned out of its position (round γ) through an angle θ, and therefore $u=\text{A}\varepsilon^{nt\sqrt{-1}}$ is an equation indicating that the motion of the point (u) results from rotation round the origin at the distance A, the angle nt being described in the time t.

V. Application of the Symbolic Forms to determine the Motion of a Rigid Body about its Centre of Gravity.

(96.) The symbolic forms $u.v$ and $u \times v$ are singularly useful, as it appears to me, in all cases of the Motion of a Rigid Body in space, especially as regards Rotation. Considerable simplification is also gained by employing $d\alpha$, $d\beta$, $d\gamma$ to denote the angular motions of the three axes α, β, γ. I shall now proceed to consider this case.

I shall take α, β, γ to denote three rectangular directions fixed in the Rigid Body, and x, y, z the coordinates of any particle (m) of the body. On this supposition x, y, z are constants as regards t, while α, β, γ are variables. The origin is the fixed point (the centre of gravity, namely,) about which the body moves. u denotes the distance of m from the origin, and therefore

$$u=x\alpha+y\beta+z\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

(97.) Now the rigidity of the body requires that the velocity $\left(\frac{du}{dt}\right)$ of m shall be at right angles to u always; this may be expressed (see art. 44) by putting

$$\frac{du}{dt}=\mathrm{D}\omega.u, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

where ω denotes some unknown line. It may be shown, as follows, that ω is a function of t *only*, or, in other words, that ω is the same for all points of the body, *i.e.* for all values of u.

Let (u') be *any point in space*, and let us assume, as we may, that this point moves always with a velocity $\mathrm{D}\omega.u'$; then

$$\frac{du'}{dt}=\mathrm{D}\omega.u'; \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

and hence, by (2),

$$\frac{d(u'-u)}{dt}=\mathrm{D}\omega.(u'-u). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

Now $u'-u$ is the line joining the two points (u) and (u'), and the square of its length is

$$(u'-u)\times(u'-u),$$

and

$$\frac{d}{dt}\{(u'-u)\times(u'-u)\}=2\frac{d(u'-u)}{dt}\times(u'-u)=0\,;$$

for (4.) shows that $\frac{d(u'-u)}{dt}$ and $u'-u$ are at right angles. Consequently the length of the line $u'-u$ is invariable.

In precisely the same way (3.) shows (what indeed is otherwise obvious) that the line u' is of invariable length.

Hence the point (u') is rigidly connected with the origin and with the point (u); and consequently (u') *is a point of the rigid body.* Therefore, comparing (2.) and (3.), it appears that ω does not vary when we pass from one point to another of the rigid body.

This result is of great importance, and furnishes, in the simplest possible manner, every formula necessary for determining the motion of the rigid body.

(98.) *The symbol ω represents, in direction,* the *instantaneous axis of rotation.* For (2.) shows that, when u coincides in direction with ω, $\frac{du}{dt}=0$; consequently all the points of the body which lie in the direction of ω are quiescent at the instant t.

(99.) *The magnitude of ω is the instantaneous angular velocity.* For, let α' denote any unit line fixed in the body at right angles to ω; then (see art. 19) $\frac{d\alpha'}{dt}$ is the angular velocity, at least in magnitude. But, by (2.), $\frac{d\alpha'}{dt}=\text{D}\omega.\alpha'$; and, since ω is at right angles to α', $\text{D}\omega.\alpha'$ has the same magnitude as ω (see art. 40.). Wherefore ω has the same magnitude as $\frac{d\alpha'}{dt}$, and therefore represents the angular velocity in magnitude.

(100.) Hence the result above obtained, namely,

$$\frac{du}{dt}=\text{D}\omega.u, \quad \ldots \quad (4.)$$

may be thus enunciated:—the rigid body is, at the time t, moving about a certain instantaneous axis, with a certain angular velocity; and if we assume ω to denote that axis, *in direction,* and the angular velocity *in magnitude,* then the velocity $\left(\frac{du}{dt}\right)$ of any point (u) of the rigid body is obtained by performing the operation ($\text{D}\omega$.) upon u.

(101.) If we put

$$\omega=\omega_1\alpha+\omega_2\beta+\omega_3\gamma, \quad \ldots \quad (5.)$$

where ω_1, ω_2, ω_3 denote numerically the projections of the line ω on the three coordinate directions α, β, γ, we find by (4.),

$$\frac{du}{dt}=\omega_1\text{D}\alpha.u+\omega_2\text{D}\beta.u+\omega_3\text{D}\gamma.u. \quad \ldots \quad (6.)$$

Hence it appears that $\frac{du}{dt}$ results from the superposition of three angular velocities ω_1, ω_2, ω_3 about the axes α, β, γ respectively: for, by (4.), $\text{D}(\omega_1\alpha).u$ (or $\omega_1\text{D}\alpha.u$) denotes a velocity of the point (u) resulting from an angular velocity ω_1 about the axis α, $\omega_2\text{D}\beta.u$ that resulting from ω_2 about β, and $\omega_3\text{D}\gamma.u$ that resulting from ω_3 about γ: and (6.) shows that the actual velocity of u is the resultant of these three velocities.

If we put for u its value $x\alpha+y\beta+z\gamma$, (6.) becomes immediately

$$\frac{du}{dt}=(\omega_1 y-\omega_2 x)\gamma+(\omega_2 z-\omega_3 y)\alpha+(\omega_3 x-\omega_1 z)\beta.$$

Whence it follows that the velocity of the point (xyz) is equivalent to the three component velocities $\omega_2 z-\omega_3 y$ parallel to x, $\omega_3 x-\omega_1 z$ parallel to y, and $\omega_1 y-\omega_2 x$ parallel to z.

(102.) The theory of the composition and resolution of angular velocities is completely expressed by (4.); for, if it be required to find the effect of the superposition of two angular velocities represented by ω and ω', we find by (4.) that the velocity produced by ω in any point (u) is $D\omega.u$, and that produced by ω' is $D\omega'.u$. The actual velocity of (u) will be the resultant of these, that is,

$$D\omega.u+D\omega'.u, \text{ or } D(\omega+\omega').u.$$

Now by (4.) $D(\omega+\omega').u$ is the effect of an angular velocity represented by $\omega+\omega'$. Hence it follows that the two angular velocities ω and ω' superposed produce the same effect as the angular velocity $\omega+\omega'$; and $\omega+\omega'$ is the third side of the triangle formed of the two lines ω and ω'.

(103.) The equation of motion of a rigid body acted on by any forces (about its centre of gravity) is easily obtained as follows.

Let U denote the accelerating force in action on m, *i. e.* at the point (u); then, by Sect. III., we have

$$\Sigma mu.\frac{d^2u}{dt^2}=\Sigma mu.\text{U},$$

which is the same thing (see art. 86) as

$$\frac{d}{dt}\left(\Sigma mu.\frac{du}{dt}\right)=\Sigma mu.\text{U}. \quad \ldots\ldots\ldots \quad (7.)$$

Now, putting for u its value (1.), and supposing that α, β, γ are the *principal axes* of the body, we find

$$\Sigma mu.\frac{du}{dt}=\left(\alpha.\frac{d\alpha}{dt}\right)\Sigma mx^2+\left(\beta.\frac{d\beta}{dt}\right)\Sigma my^2+\left(D\gamma.\frac{d\gamma}{dt}\right)\Sigma mz^2.$$

But, by (4.) and (5.),

$$\frac{d\alpha}{dt}=D\omega.\alpha=-\omega_2\gamma+\omega_3\beta;$$

$$\therefore \quad D\alpha.\frac{d\alpha}{dt}=\omega_2\beta+\omega_3\gamma^*.$$

Similarly, $$D\beta.\frac{d\beta}{dt}=\omega_3\gamma+\omega_1\alpha,$$

and $$D\gamma.\frac{d\gamma}{dt}=\omega_1\alpha+\omega_2\beta;$$

whence, introducing D, we find

$$\Sigma m Du.\frac{du}{dt}=(\omega_2\beta+\omega_3\gamma)\Sigma mx^2+(\omega_3\gamma+\omega_1\alpha)\Sigma my^2+(\omega_1\alpha+\omega_2\beta)\Sigma mz^2=\text{A}\omega_1\alpha+\text{B}\omega_2\beta+\text{C}\omega_3\gamma,$$

where A, B and C denote respectively $\Sigma m(y^2+z^2)$, $\Sigma m(z^2+x^2)$, $\Sigma m(x^2+y^2)$.

Thus (7.) becomes

$$\frac{d}{dt}(\text{A}\omega_1\alpha+\text{B}\omega_2\beta+\text{C}\omega_3\gamma)=\Sigma m Du.\text{U}. \quad \ldots\ldots\ldots \quad (8)$$

* Or thus: $D\alpha.\frac{d\alpha}{dt}=D\alpha.(D\omega.\alpha)=-(D\alpha.)^2\omega=\omega-(\omega\times\alpha)\alpha=\omega_1\alpha+\omega_2\beta.$

This is the general equation of motion of a rigid body about a fixed point. It gives the three well-known equations immediately by equating the coefficients of α, β, γ*. But the equation (8.) as it stands is more available for the solution of problems, and furnishes results much more simply, than the three equations alluded to. Along with (8.) we must employ the equation

$$\frac{du}{dt}=\mathrm{D}\omega . u. \qquad (9.)$$

And these two are completely equivalent to the six equations commonly employed.

(104.) The line represented by the symbol $\mathrm{A}\omega_1\alpha+\mathrm{B}\omega_2\beta+\mathrm{C}\omega_3\gamma$ has a remarkable relation to the instantaneous axis $\omega_1\alpha+\omega_2\beta+\omega_3\gamma$, which may be thus interpreted. Suppose the rigid body to undergo a *distortion* or *unequal expansion* of such a nature, that all lines *in it* parallel to α become A times longer than before, all lines parallel to β, B times longer, and all lines parallel to γ, C times longer. The effect of this will be to convert the unit α into $\mathrm{A}\alpha$, β into $\mathrm{B}\beta$, γ into $\mathrm{C}\gamma$; and thus the line $\omega_1\alpha+\omega_2\beta+\omega_3\gamma$ will be converted into $\mathrm{A}\omega_1\alpha+\mathrm{B}\omega_2\beta+\mathrm{C}\omega_3\gamma$. This latter line, therefore, I may call the *Distorted Instantaneous Axis.*

(105.) The *distortion* here alluded to is one of great importance to be noted, because it indicates an operation which has immediate connection with many remarkable physical phenomena as well as with various theories in Solid Geometry. As regards its geometrical meaning, if we conceive the rigid body to be a solid composed of spherical shells having a common centre at the origin, each shell will be converted into an ellipsoid by the distortion. The sphere whose radius is unity will be changed into an ellipsoid whose axes are $\mathrm{A}\alpha$, $\mathrm{B}\beta$, $\mathrm{C}\gamma$; and the axes of the other ellipsoids will be parallel and proportional to these.

The line represented by the symbol

$$\mathrm{A}\alpha+\mathrm{B}\beta+\mathrm{C}\gamma$$

is an important determining element. If we assume ω' to denote what ω becomes in consequence of the distortion, it may be easily seen that ω' is a *distributive function* of ω and $\mathrm{A}\alpha+\mathrm{B}\beta+\mathrm{C}\gamma$; and from this fact a number of curious symbolical relations may be deduced. But I must not dwell upon this subject of distortion now further than my immediate purpose requires.

(106.) Using ω' for brevity to denote the *distorted instantaneous axis,*

$$\mathrm{A}\omega_1\alpha+\mathrm{B}\omega_2\beta+\mathrm{C}\omega_3\gamma,$$

I may observe that the equation (8.), that is,

$$\frac{d\omega'}{dt}=\Sigma m\mathrm{D}u . \mathrm{U}, \qquad (10.)$$

* Observing that $\frac{d\alpha}{dt}=\mathrm{D}\omega.\alpha$, $\frac{d\beta}{dt}=\mathrm{D}\omega.\beta$, $\frac{d\gamma}{dt}=\mathrm{D}\omega.\gamma$, we find the coefficient of γ in the first member to be

$$\mathrm{C}\frac{d\omega_3}{dt}+(\mathrm{B}-\mathrm{A})\omega_1\omega_2;$$

and putting $\mathrm{U}=\mathrm{X}\alpha+\mathrm{Y}\beta+\mathrm{Z}\gamma$, we find, in the second member,

$$\Sigma m(x\mathrm{Y}-y\mathrm{X}).$$

gives the velocity of the point (ω') *in space*, the differential letter, d, denoting *absolute* change of position. But it is often important to determine the motion of this point *relatively to the rigid body*; and this may be done as follows:—

Let us assume δ to denote *relative* change of position with reference to the rigid body; then it is evident that

$$\frac{d\omega'}{dt}=\frac{\delta\omega'}{dt}+\mathrm{D}\omega.\omega'; \quad \ldots \ldots \ldots \quad (11.)$$

for the velocity of the point (ω') is the resultant of two velocities, namely, that relative to the body, and that arising from the motion of the body; the former is $\frac{\delta\omega'}{dt}$, and the latter, by (9.), is $\mathrm{D}\omega.\omega'$.

Hence, and by (10.), we find

$$\frac{\delta\omega'}{dt}=\Sigma m\mathrm{D}u.\mathrm{U}-\mathrm{D}\omega.\omega'. \quad \ldots \ldots \ldots \quad (12.)$$

This equation gives the velocity of the point (ω') *relatively to the rigid body.*

Now it is clear that if we can solve (10.) and (12.) the motion of the rigid body is determined; for we shall then know (by (12.)) how the line ω' moves *in the rigid body*, and by (10.) how ω' moves *in space*; and thus, by the intervention of ω', we shall obtain the motion of the rigid body in space.

(107.) As an example of this I shall take the case of the earth attracted by the sun, and point out briefly how (10.) and (12.) determine the motion of the polar axis. In this case $\mathrm{A}=\mathrm{B}$, and $\mathrm{C}=(1+\lambda)\mathrm{A}$, when λ is a small number: also the instantaneous axis ω very nearly coincides in direction with the polar axis γ. Hence, and by art. 89, we have

$$\omega'=\mathrm{A}(\omega+\lambda\omega_3\gamma)$$

$$\Sigma m\mathrm{D}u.\mathrm{U}=\mathrm{A}\frac{3m'}{r'^5}\mathrm{D}u'.(u'+\lambda z'\gamma).$$

Here u' denotes (symbolically) the sun's distance, r' is the magnitude of u', m' the sun's mass.

Thus, observing that $\mathrm{D}u'.u'$ and $\mathrm{D}\omega.\omega$ are zero, (10.) and (12.) become

$$\frac{d\omega}{dt}+\lambda\frac{d(\omega_3\gamma)}{dt}=\lambda\frac{3m'}{r'^5}z'\mathrm{D}u'.\gamma$$

$$\frac{\delta\omega}{dt}+\lambda\frac{\delta(\omega_3\gamma)}{dt}=\lambda\frac{3m'}{r'^5}z'\mathrm{D}u'.\gamma-\lambda\omega_3\mathrm{D}\omega'.\gamma.$$

In the terms multiplied by λ we may approximate on the supposition that γ is fixed and $\omega=n\gamma$, where n is the earth's angular velocity about its polar axis. This reduces the two equations to

$$\frac{d\omega}{dt}=\lambda\frac{3m'}{r'^5}z'\mathrm{D}u'.\gamma. \quad \ldots \ldots \ldots \quad (13.)$$

$$\frac{\delta\omega}{dt}=\lambda\frac{3m'}{r'^5}z'\mathrm{D}u'.\gamma. \quad \ldots \ldots \ldots \quad (14.)$$

Now let us take new directions $(\alpha', \beta', \gamma')$ so, that γ' shall coincide for a moment with the polar axis γ, while the plane $(\alpha'\gamma)$ contains the sun's distance u'; then

$$u'=x'\alpha'+z'\gamma$$

$$\mathrm{D}u'.\gamma=-x'\beta';$$

wherefore (14.) becomes

$$\frac{\delta\omega}{dt}=\left(-\lambda\frac{3m'}{r'^5}x'z'\right)\beta'.$$

The coefficient of β' here, being multiplied by λ, may be regarded as invariable during one day, inasmuch as x' and z' take a year to go through their values: also, since δ implies that the earth is considered as fixed, the sun, and therefore the direction β', must be supposed to revolve about γ from east to west, through 360° in the day. The velocity $\frac{\delta\omega}{dt}$ therefore is constant in magnitude but changes its direction (which is always perpendicular to γ) uniformly through 360° in the day. It appears therefore that the point (ω) describes a daily circle, and therefore the line ω describes a daily cone about γ. From this it follows (observing that ω and γ make a very small angle with each other), that the mean daily angular motion in space of γ and that of the direction of ω (manifestly $\frac{\omega}{n}$ very nearly) are identical. Wherefore

$$\frac{d\gamma}{dt}=\frac{1}{n}\frac{d\omega}{dt}=\frac{\lambda}{n}\frac{3m'}{r'^5}z'\mathrm{D}u'.\gamma, \text{ by (13.)};$$

or, since

$$z'=u'\times\gamma,$$

$$\frac{d\gamma}{dt}=\frac{\lambda}{n}\frac{3m'}{r'^5}(u'\times\gamma)\mathrm{D}u'.\gamma. \qquad (15.)$$

If now we assume γ' to be at right angles to the plane of the ecliptic, and α' to point towards the first point of Aries, we have

$$u'=r'(\alpha'\cos n't+\beta'\sin n't),$$

where $n't$ is the sun's longitude, n'^2 being $\frac{m'}{r'^3}$.

Wherefore, observing that γ is at right angles to α', we find

$$\frac{d\gamma}{dt}=\frac{3n'^2\lambda}{n}(\gamma\times\beta')\sin n't\,(\cos n't\mathrm{D}\alpha'.\gamma+\sin n't\mathrm{D}\beta'.\gamma).$$

If we integrate this between the limits 0 and $\frac{2\pi}{n'}$, we find the *annual variation of* γ, which therefore is

$$\frac{3n'\lambda\pi}{n}(\gamma\times\beta')\mathrm{D}\beta'.\gamma.$$

This represents in magnitude and direction the actual space described by the point (γ) in one year, *i. e.* the angular motion of the pole, or the precession. If ϖ denote the obliquity of the ecliptic,

$$\gamma=\gamma'\cos\varpi+\beta'\sin\varpi,$$

and
$$\therefore \gamma\times\beta'=\sin\varpi$$
$$D\beta'.\gamma=\alpha'\cos\varpi.$$

Wherefore the annual precession of the pole is

$$\left(\frac{3n'\lambda\pi}{n}\cos\varpi\sin\varpi\right)\alpha',$$

α' indicating that its direction is perpendicular to the solstitial colure, and retrograde as regards the sun's motion.

I have gone through this example because in every step it shows the use of the symbolic forms.

VI. Application of the Symbolic Forms to Physical Optics.

(108.) The use of the symbolic forms $u.v$ and $u\times v$ in Physical Optics is very remarkable; but as this paper is already so long, I can only just allude to the subject.

In the Transactions of the Cambridge Philosophical Society I have shown (in a paper read March 17, 1847), that if v denote the displacement at the point (u) of an uncrystallized medium, where

$$u=x\alpha+y\beta+z\gamma$$
$$v=\xi\alpha+\eta\beta+\zeta\gamma,$$

and if A and B be two constants (namely, the coefficients of direct and transverse elasticity); then

$$\frac{d^2v}{dt^2}=B\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)v+(A-B)\left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}\right)\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right).$$

And this result I obtained by merely considering the *disarrangement* of the medium about the point u, without any assumption respecting the constitution of the medium, except that it possessed direct and transverse elasticity.

Now if we employ the letter Ω to denote the operation

$$\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz},$$

the above equation, by the aid of the symbolic forms, immediately assumes the simple form

$$\frac{d^2v}{dt^2}=A\Omega(\Omega\times v)-B(D\Omega.)^2v. \quad \ldots\ldots\ldots \quad (1.)$$

The symbol Ω has a very important signification when written before any quantity, U, which is a function of x, y and z: for the *direction* of the line ΩU is that direction *perpendicular to which there is no variation of* U; while the magnitude of ΩU is the *rate of variation* of U *in that direction, i. e.* as we pass from point to point of the medium in the direction of ΩU.

Again, $\Omega\times v$ denotes the *rarefaction* of the medium, at the point (u), resulting from the displacements represented by v; while $\Omega.v$ denotes, *in magnitude and plane*, the *lateral disarrangement* of the medium.

It is clear, therefore, that the symbolic forms $u.v$ and $u\times v$ must be of great use in Physical Optics; indeed the facility they give of following out investigations respecting undulatory movements is so great, that the whole subject of reflexion and refraction, in crystallized as well as in uncrystallized media, and the mathematical explanations of the phenomena connected with polarization, double refraction, &c., may be reduced to a state of simplicity which could hardly be expected in such a difficult subject.

(109.) In the paper above alluded to, I obtained also the equation of vibratory motion generally, for any crystallized medium, without any of those assumptions which mathematicians have found it necessary to make in order to render the investigation manageable; especially, without assuming the vibrations of a plane polarized ray to be *in* the plane of polarization, which appears to me to be a highly objectionable assumption. By the aid of the symbolic forms, the general equation of vibratory motion, where the transmission of transverse vibrations is possible, is thus expressed:

$$\frac{d^2v}{dt^2}=\left(A_1\alpha\frac{d}{dx}+A_2\beta\frac{d}{dy}+A_3\gamma\frac{d}{dz}\right)(\Omega\times v)$$
$$+D\Omega.\left\{\left(B_2\frac{d\eta}{dz}-B'_3\frac{d\zeta}{dy}\right)\alpha+\left(B_3\frac{d\zeta}{dx}-B'_1\frac{d\xi}{dz}\right)\beta+\left(B_1\frac{d\xi}{dy}-B'_2\frac{d\eta}{dx}\right)\gamma\right\}. \quad . \quad . \quad (2.)$$

Here A_1, A_2, A_3 are coefficients of *direct elasticity*, corresponding to A in equation (1.); and B_1, B_2, B_3, &c. are six coefficients of *transverse elasticity*, corresponding to B in (1).

FRESNEL's hypothesis, that the vibrations of a plane polarized ray are *perpendicular* to the plane, makes

$$B_1=B'_1, \qquad B_2=B'_2, \qquad B_3=B'_3,$$

while the hypothesis, that the vibrations are *in* the plane of polarization, makes

$$B_1=B_2, \qquad B_2=B'_3, \qquad B_3=B_1.$$

On the former hypothesis the equation becomes

$$\frac{d^2v}{dt^2}=\left(A_1\alpha\frac{d}{dx}+A_2\beta\frac{d}{dy}+A_3\gamma\frac{d}{dz}\right)(\Omega\times v)-(D\Omega.)^2(B_1\xi\alpha+B_2\eta\beta+B_3\zeta\gamma); \quad . \quad . \quad . \quad (3.)$$

while, on the latter, it becomes

$$\frac{d^2v}{dt^2}=\left(A_1\alpha\frac{d}{dx}+A_2\beta\frac{d}{dy}+A_3\gamma\frac{d}{dz}\right)(\Omega\times v)$$
$$-(D\Omega.)\left\{B_1\left(\frac{d\eta}{dz}-\frac{d\zeta}{dy}\right)\alpha+B_2\left(\frac{d\zeta}{dx}-\frac{d\xi}{dz}\right)\beta+B_3\left(\frac{d\xi}{dy}-\frac{d\eta}{dx}\right)\gamma\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

For transverse vibrations the rarefaction $(\Omega\times v)$ (see above) is zero, which further simplifies (3.) and (4.). By equating the coefficients of α, β, γ in (4.), thus simplified, we obtain MACCULLAGH's three equations. The equation (3.) coincides in every way with FRESNEL's theory.

The conception of *distortion* alluded to in Sect. V. applies to (3.) in a very remarkable manner; for if we put

$$A_1\alpha\frac{d}{dx}+B_1\beta\frac{d}{dy}+C_1\gamma\frac{d}{dz}=\Omega',$$

$$B_1\xi\alpha+B_2\eta\beta+B_3\gamma\zeta=v',$$

then Ω' is Ω *distorted* by the line $A_1\alpha+A_2\beta+A_3\gamma$; and v' is v *distorted by* the line $B_1\alpha+B_2\beta+B_3\gamma$. In using the expression "distorted by," I anticipate a signification which I hope to explain in a future paper; but the fact is, Ω' is a *distributive function* of Ω and $A_1\alpha+A_2\beta+A_3\gamma$ (see art. 105), and therefore, symbolically, Ω' is Ω *multiplied by* $A_1\alpha+A_2\beta+A_3\gamma$.

Now the equation (3.) becomes

$$\frac{d^2v}{dt^2}=\Omega'(\Omega.v)-(D\Omega.)^2v',$$

or, for transverse vibrations, simply

$$\frac{d^2v}{dt^2}=-(D\Omega.)^2v'.$$

But I must reserve the consideration of this remarkable equation, merely remarking here that it shows very clearly how the force brought into play by the disarrangement of a medium resulting from transverse vibrations is altered by the crystallization. In the uncrystallized medium the force is, by (1.),

$$-(D\Omega.)^2(B\xi\alpha+B\eta\beta+B\zeta\gamma),$$

and in the crystallized medium it is

$$-(D\Omega.)^2(B_1\xi\alpha+B_2\eta\beta+B_3\zeta\gamma).$$

In one case it is found by performing the operation $-(D\Omega.)^2$ on v, in the other by performing the same operation on v *distorted* as by the threefold expansion of the medium, consisting of a drawing out in the direction α in the proportion of B_1 to B, in the direction β, of B_2 to B, and in the direction γ, of B_3 to B.

Upper Norwood, Surrey,
April 19, 1852.